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DOTTORANDO
ANDREA DEGHI

TUTOR
PROF. PAOLO PIN

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To Yiming, for bringing out the best in me.

Abstract

“..before the 2007-2008 crisis, the thrust of economic discussion was that diversification, or interconnectedness, was a great thing. However, the belief that diversification enables risk to be spread was proven wrong during the crisis. The high degree of interconnectedness in the financial system facilitated the breakdown and became part of the problem,”

Nobel Laureate Professor Joseph Stiglitz

Keynote Address, IMF Networks Conference May 8, 2014

Indeed, as the events of the 2007 Crisis unfolded, it was clear that the failure or even rumors about the failure of one single institution could trigger freezes in numerous capital markets and widespread default in other financial institutions. How this was brought about, however, was everything but clear. Ten years later, as we stand today, the literature has progressed but many questions remain unresolved.

The first question at hand is of course how banks were related and how these bilateral relationships were able to act as a passage of contagion. On the liability side, borrowing between banks provides liquidity insurance against their idiosyncratic shocks. In bad times however, insurance networks malfunctioned, and more centrally connected banks were able to monopolize on their market power to secure a larger surplus in the scramble for liquidity. My first paper further explores the relationship between network position and the ability to hoard liquidity.

In addition, assets on bank balance comprise another important channel of contagion and one should ask to which extent cross holding of asset portfolios is optimal. When does the benefit from diversification over idiosyncratic risks dominate? And when does market risk increase systemic risk of common portfolio holdings? The second essay analyzes these questions from the perspectives of private and social welfare.

In the end, one must also wonder how these networks were formed at the beginning and how the endogenous formation correlates with the structural implications. The third paper takes a step back and begins with a world where banks optimally choose links, prices and the amounts of trade. This endogenously determined network structure then serves as a coherent laboratory for understanding various frictions in the interbank market such as market freezes.

Financial networks are complex and so is the research about them. Hence, in this thesis, I have attempted to shed light on them from various angles, utilizing both bilateral network data as well as theoretical analytical tools. It is my hope that taken together, this set of essays can contribute to a holistic understanding of interconnectedness in financial market. Below I describe the individual papers in more details.

1 Chapter: Liquidity Hoarding and Core-Periphery Market Structure

Abstract

The paper empirically investigate the importance of liquidity hoarding in the Italian overnight interbank market during the financial crisis of 2008. We show that in the aftermath of Lehman shock, lending behaviour in the interbank market becomes more sensitive to banks' network position. Viewing the network from a core-periphery perspective, core banks in the system decreased lending to periphery banks and charged higher prices during a time of liquidity shortage in the financial crisis. We further verify the driver of this wedge to be difference in market power induced by network structure. These results provide a coherent explanation for the mixed views on liquidity hoarding provided in the empirical literature. Overall, this sheds light on the risks of the financial system's interconnectedness as a source of market fragility and illiquidity.

1.1 Introduction

The global financial crisis in 2008 highlighted the key role of the intertwined nature of financial markets in shaping the transmission of risk and the buildup of fragility. In particular, the failure of Lehman Brothers generated a financial contagion that destabilized many banks without a direct exposure.

The theoretical literature has suggested three main channels for why such interbank disruptions might occur. The first stream relies on the presence of asymmetric information in driving liquidity freezes. In Flannery (1996), Freixas and Jorge (2008) and Heider, Hovrova, and Holthausen (2009), adverse selection causes unraveling of the lending chain as uncertainty in credit rises with the the fraction of risky investors. Another set of papers

acknowledge counterparty risk under completely informed agents. Furfine (2001), Flannery (1996) and Bruche and Gozales-Aguado (2010) suggest that as credit risk increases, funding costs become excessively high, deterring access to interbank markets. Finally, even in the absence of counterparty risk considerations, banks may hoard liquidity in market wide stress events. As Diamond and Rajan (2009) , Allen, Carletti, and Douglas (2009) and Caballero and Krishnamurthy (2008) point out, banks would like to hoard liquidity in order to insure themselves against and/or to make profits from lending at higher prices when prices surge in the future.

Our paper focuses on and extends the third channel - liquidity hoarding. We show that liquidity hoarding was present in the Italian overnight interbank lending market but that the ability to hoard liquidity is highly correlated with banks' market power as determined by their network position.

The importance of market power on bank interest margin in the real economy has been widely studied. Berlin and Mester (2015) finds a relationship between banks' liability structure and the average loan rate. When a bank has monopoly power, it can offer a firm a lower-than-competitive rate early in the firm's life and then make up for this by clearing a higher-than-competitive rate later in the firm's life. The authors measure market power with the ratio of core deposits providing evidence that banks with high core deposit ratios might react to an increase in credit risk by tightening their credit screens more than banks with lower core deposit ratios. In a similar vein, Petersen and Rajan (1995) also show that when a bank has monopoly power, it can offer a firm a lower-than-competitive rate early in the firm's life and then make up for this by charging a higher-than-competitive rate later in the firm's life.

We adapt market power to the interbank market by introducing the notion of a core periphery as a suitable measure. In simple terms, it is defined as a connected network that has two tiers, a core and a periphery, the core composed by a sample of banks fully connected to each other, whereas peripheral banks are only connected to the core.

We chose this measure for numerous reasons. On one hand, it is a global concept taking

the entire network structure into consideration. On the other hand, among global statistics, the core periphery definition is the most persistent over time and has been widely observed as robust in a number of countries. For instance, Craig and Von Peter (2014) shows the existence of a core-periphery structure in the German interbank. Fricke and Lux (2014) identify a core stable over time with a high persistence of the banks' positions and member of the core or periphery. This structure is also consistent with empirical evidence on intermediation in several markets, including the federal funds market (Afonso, Kovner, and Schoar (2011) and Bech and Atalay (2010)), international interbank markets (Boss, Elsinger, Martin, and Thurner (2004) for Austria; Van Lelyveld and in t' Veld (2012); Di Maggio, Kermani, and Song (2016) in the corporate bond markets. Hence, the core periphery structure is a prominent feature in reality so that the distinct cut off between the two tiers must bear significant economic implications. ¹

Indeed, after establishing that the Italian overnight interbank market demonstrates a persistent core periphery structure, we find that the cutoff in connectivity correlates core banks with cheaper access to funding and higher return on lending relative to their periphery counterparts. Utilizing the shock to interbank markets of the recent financial crisis, which triggered an increase in the expectation of future funding cost and hence the incentive to hoard liquidity.

Our results show that being in the periphery in the aftermath of the financial crisis leads on average 10bps higher prices paid by banks in the periphery vis and more than 50bps difference in the average interest rate received from lending transactions a vis banks in the core. This shows that being in the core versus periphery bears economically and statistically significant effects on funding costs and lending rates. To provide further evidence that the mechanism at play is market power, we allow for a finer categorization within the core and show that core banks with a larger number of bargaining counterparties are able to

¹Nevertheless, core-periphery structure does not emerge randomly. Financial relations are formed consciously by financial institutions. Given limited scope of a paper, we will abstract away from network formation and take the network of financial interconnections as exogenously fixed. Instead, we focus on the core-periphery as a suitable way to measure market power in the interbank network and how it can affect liquidity provision in crisis time.

obtain a larger share of the surplus through cheaper borrowing and higher return on lending, consistent with a Nash bargaining setting with renegotiable contracts as in Stole and Zwiebel (1996) or updating outside options as in Duffie and Wang (2016).

Distinct heterogeneity in the ability to hoard may have been why the empirical literature on liquidity hoarding finds mixed results. Afonso and Lagos (2015) do not find evidence for liquidity hoarding in the market post the Lehman Brothers bankruptcy. Acharya and Merrouche (2010) show that in the UK interbank market during the financial crisis, riskier banks hold more reserves relative to the expected payment value and that borrowing rates became independent from bank characteristics. The authors interpret these results as evidence as precautionary liquidity hoarding. Furfine (2002) reports that despite the Russia's effective default on its sovereign bonds and the nearly collapse of LTCM, banks did not hoard liquidity and the market and continued to channel liquidity to the banks in most need. In the Italian interbank market, Angelini, Nobili, and Picillo (2011) find that more liquid lenders charge higher rates after August 2007 but the economic magnitude of their estimate is small. Overall, we stress that the tiered structure of the market leads to differential capacities to hoard liquidity, leading to important distributional effects.

Of course, one might be concerned about other confounding channels. However, note that core banks can borrow at cheaper costs relative to periphery banks post crisis. This challenges the significance of counterparty risk explanations because the recent crisis was centered in capital markets in which mostly big banks in the core participated in. In other words, if counterparty risk was the determinant of interbank market freezes, we should have instead observed a hike in credit premium for core banks who were directly exposed. Although the presence of a small amount of credit risk in the core would only strengthen the magnitude of our estimates, we aim for a cleaner identification by removing banks prone to direct balance sheet exposures including large banks and foreign banks. The economic and statistical robustness of our results in this subset further rules out surge in funding costs due to counterparty risk as the main driver of interbank frictions.

In most of the paper, we focus on events unfolding in the crisis because it serves as a shock

to render the effects of bargaining power more pronounced - when aggregate conditions are deteriorating, the incentive to hoard liquidity is larger relative to times of excess liquidity, and market power is able to claim a larger fraction of the pie in the scramble for liquidity. However, interbank frictions of bargaining and market power are also present in normal times and we hope that our results offer insight on their general existence. Both are important for informing policy. In the words of Donald Kohn, the former vice chairman of the Federal Reserve Board: “Supervisors need to enhance their understanding of the direct and indirect relationships among markets and market participants, and the associated impact on the system. Supervisors must also be even more keenly aware of the manner in which those relationships within and among markets and market participants can change over time and how those relationships behave in times of stress”². In the concluding section, we detail policy implications of our results and suggest potential extensions as more data becomes available.

The remainder of this paper is organized as follows: Section 1.2 introduces the Italian e-MID interbank data; Section 1.3 demonstrates the clear core-periphery structure of e-MID interbank, which shapes trading behaviour. Section 1.4 describes the main results on the importance market structure liquidity hoarding at times of financial distress. Section 1.5 concludes.

1.2 The Italian Interbank Market

On the e-MID platform transaction with different maturities are performed, but overnight transactions represent the overwhelming majority of the interbank market. The dataset is rich and for each one of the transactions, we have information about the date and time of the trade, the volumes traded (in millions of Euro), the price (in annual percentage), the identifier of the quoting and ordering bank. Furthermore we have an indicator variable for the balance sheet size of the banks, which follows Bank of Italy classification in five different groups according to their weighted asset portfolio for Italian banks and an indicator variable

²Senate testimony, June 5, 2008

international banks operating on e-MID platform³. The set of variables we have for each one of the transactions occurred in the period from January 1 1999 to September 30 2009, permits us to do an accurate daily analysis of the lending practices in the market and how these change over time.

The overnight rate is bounded between the rate of the marginal lending facility and the rate of the overnight deposit facility. Banks trade in the market in order to fulfill their reserve requirements over the maintenance period imposed by ECB. Gabrieli (2011) showed that the spread of the policy interest rates has been found to be liquidity short before the crisis, while after August 2007 it became liquidity long. Furthermore, the *ON* interest rate indicated an important increase in volatility, after the two Bearn Stearns collapse, which reduced again until the failure of Lehman Brothers, which widened again the dispersion of interest rates. These trends indicate that the financial crisis impacted significantly the loans terms in the market, altering the capacity of banks to access liquidity from the interbank market. Figure 1 shows the average total volumes traded over the year and the number of *ON* and *ONL* trades on the e-MID market. From the graph it clearly appears that since the beginning of the financial crisis, both the number of transactions, and their value have constantly reduced over time. Assuming that the liquidity needs of banks were unchanged over such a short time period, it could be argued that either these institutions resorted to other forms of capital, or they had a serious lack of short-term capital for their operations.

Similarly, in 2009 the share of banks borrowing liquidity dropped to less than 80%, while during the period between 1999 and 2008 oscillated between 88% and 92%. The new pattern becomes clearer if we consider also the dynamics of the fractions of banks both borrowing and lending capital, which from a stable average of 88% until 2008, declined to less than 70% of the total of the active banks (Figure 2). These results seem to indicate that the e-MID is a market where agents with dominant roles interact each other. Additional evidence on this point is provided by the high dispersion of the deposits (both on the lending and borrowing side) with respect to the average, which confirms the presence of heterogeneity in the trading

³See Tables 3 and 4 for general statistics on the size distribution of banks' population

behaviour of banks. In this regard, Affinito (2012) reports a positive relation between spreads measures and foreign banks operating in the market, arguing that risk perception toward these banks raised in the aftermath of the crisis. Angelini, Nobili, and Picillo (2011) finds however that such effect is not economically significant, despite banks may have become more sensitive to credit quality of their counterparties and charging higher interest rates to borrowers with lower credit standing. In particular, after the Lehman collapse market players may have reduced greatly the faith in the too-big-too fail implicit warranty, leading to a decline in the discount large banks were enjoying before the start of the financial crisis.

Finally, examining the change in the flows of liquidity, we report high concentration of the liquidity on both the lending and borrowing side of the market. Less than 8% of the active banks in the market accounted for more than 40% of the total liquidity exchanged in 2006. Despite the percentage slightly increased in the following years, the evidence shows how the liquidity is concentrated on a handful of banks operating in the market.

1.3 Market Structure Identification

In this section, we define the variables that we use in the analysis and discuss the empirical strategy used to identify the effects of market structure on interbank liquidity and its relation with bank market power. Given our focus debunking the role played by market structure in particular during crisis time, in the remainder of the paper we will restrict the data sample to the periods starting March 1, 2006 to September 30, 2009. In particular, we discuss how we identify a particular source of market power in the interbank market on reducing asymmetric information about credit risk during tranquil times as compared to with effects that bank pair relationships have in mitigating search frictions, but at the risk of increase liquidity hoarding by market players enjoying this market power in period of crisis. We use daily transactions of the Italian e-MID from 1 April 2006 to 30 September 2009. To understand the impact of the financial crisis on the Italian interbank market, we study the period surrounding the bankruptcy of Lehman Brothers. and divide the sample in three subsample The first one starting the quarters before the two Bear Stearns's Hedge fund

bankruptcy (April 1, 2006 to June 30, 2007); the second one indicating the run up of the crisis (July 1, 2007 to September 30, 2008) and the last one representing the period after the crisis. In order to construct representative measures of network core-periphery structure we use quarterly data⁴.

As our first step for studying the market structure in e-MID overnight market, we use the sequential optimization algorithm developed by Craig and Von Peter (2014) to estimate the core of the network⁵. Craig and Von Peter (2014) show that core banks tend to be the larger banks and that core it is important for providing interbank lending to the banks in the periphery. 1 shows a negative trend in the absolute size of the cores over time, but this is not surprising given that the number of active Italian banks has been decreasing over time. Comparing the number of nodes per quarter in the Table, we can also denote the existence of a structural break in the core sizes after the quarter of Lehman collapse, with the trend going back towards its initial level in the period after the Lehman brother collapse. As suggested by Iori, De Masi, Precup, Gabbi, and Caldarelli (2008), the structural break can be likely be related either to market interest rate becoming negative or by the increased injections of liquidity of ECB in the interbank aimed to support economic growth⁶.

Despite the decreasing number of core, we can show that the composition of the core membership remained stable over time. To this aim, we calculated, we calculated the transition matrix for each quarter. The elements in Table 2 represent the frequency with which core banks move to the periphery over time. The third state (outside the sample) refers to the banks in the following quarter were not active anymore in the market. Values on the diagonal are close to unity, confirming in this way that banks tend to remain in the same tier (core or periphery), and hence that despite the high daily fluctuation in the market, the

⁴Iori, Mantegna, Marotta, Micciche, Porter, and Tumminello (2015) show that for longer aggregation periods network pattern are less volatile

⁵See section 1.6 for more details on the methodology adopted

⁶As a robustness check, for the validity of the fitting we measures also the number of errors following using the error score used in Craig and Von Peter (2014). To compare our following results using different coreness measure, we run the algorithm on the giant component of the network of active banks calculated for each quarter in the sample and assigning periphery status for the banks falling outside of the giant component for that specific quarter. Despite results show a less degree of fitting with respect to Craig and Von Peter (2014), the number of normalized errors remain constant over time around 5% of the total number of links.

composition of banks in the core remained stable over time.

To support our findings, we fitted the Italian interbank network with a continuous model of core-periphery centralization. The level of CP-centralization of the network for the whole time span and a continuous measure of coreness for each bank. This is based on an iterative algorithm based on a Markov chain model that yields an overall network centralization index for the core-periphery profile⁷. The closer is the indicator to one, the higher is the fitting of the core-periphery structure⁸. Table 1 shows that the CP-centralization indicator is on average 0.85 displaying therefore a high level of fitting of the core-periphery structure. We find that the core banks are significantly larger and more active than periphery banks. Borrowing activity is the more relevant aspect of core banks' participation in the market if we consider all the years in the aggregate. Furthermore, the financial turmoil had a substantial impact on the goodness of fit of discrete CP model on a yearly basis. Conversely, the continuous CP model based on the random walk approach, seems to achieve better levels of goodness in fitting the core of the network. These findings support the idea that we have identified a truly structural feature of the inter-bank market, which is persistent over time.

Our results complement other empirical studies on the e-MID (Gabrieli (2011), Affinito (2012), Iori, Gabbi, Germano, Hatzopoulos, Kapar, and Politi (2014), Iori, Mantegna, Marotta, Micciche, Porter, and Tumminello (2015)) that show the existence of preferential lending relationships and the impact of centrality measure on banking behavior.

⁷See Della Rossa, Dercole, and Piccardi (2013) for technical details on the methodology. The algorithm proposed by the authors also provides us with a graphical representation of the core centralization that illustrates

⁸Figures 3 and 4 give a graphical representation of the core-periphery profile. Similar to the representation of the Lorenz curve with the two corners of the graph in this case representing the complete network (top left) vs a pure star network (bottom right). Hence, the network will be considered to be more centralized the closer the curve will be to the star network profile.

1.4 Empirical Approach

1.4.1 Summary Statistics

We use daily transactions of the Italian e-MID from 1 April 2006 to 30 September 2009. To understand the impact of the financial crisis on the Italian interbank market, we study the period surrounding the bankruptcy of Lehman Brothers. and divide the sample in three sub-sample The first one starting the quarters before the two Bear Stearns's Hedge fund bankruptcy (April 1, 2006 to June 30, 2007); the second one indicating the run up of the crisis (July 1, 2007 to September 30, 2008) and the last one representing the period after the crisis.

On the basis of the last core-periphery identification, we split furthermore the sample of transaction on the basis of the type of link, and compare the differences in the distribution of the amount and rate. Comparing the results in the three different phases of the crisis, we see that relative to periphery banks, core banks were able to charge the highest prices in the transactions profits after Lehman collapse (Figure 9) and were maintained flows of liquidity in the core to core transactions constant through time. Conversely, comparing the mean and median distribution of transactions amount (Figures 5 and 6), despite core to periphery transactions accounted for the highest average value of transactions, the flows with the periphery steadily decrease in number and size in after the Lehman bankruptcy. Results remain unchanged considering the upper and lower quartile of the amount distributions (Figures 7 and 8 respectively)

To assess whether some banks are able to borrow or lend money at a better rate than others, we define the average daily credit spread as the difference between the market weighted interest rate and bank pair interest rate at which two banks transaction in one day weighted for the volumes of the transactions. Since during the same day a bank may act both as a lender and a borrower, the credit spread so defined provides a measure of the ability of a bank to borrow or lend at competitive rates relatively to the mean rate observed in that day in the same market. Figure 11 shows the distribution of the median values of the amount

traded in e-MID aggregated on monthly basis and divided by link type. As our main hypothesis suggest, transaction spread for core to core links and in particular periphery to core links shows the lowest level of spread after the Lehman collapse, whereas spread for core to periphery transaction spikes after 15 September 2008. Also in this case the results are consistent if we plot the same graph for the mean and quartiles of spread series. The pattern is confirmed looking at the cumulative distribution of profit margins for the different types of counterparties. Figure 13, shows indeed between core and periphery it exists a substantial heterogeneity between spreads between borrowers in the core and borrowers in the periphery for the period July 2008 to September 2009.

Evidence suggest hence that after the crisis, core banks, while maintaining the flows of liquidity within the core, started to restrict flows of liquidity to the periphery and charged high price for the banks seeking for liquidity. In previous studies there has been a considerable effort for explaining such variation pointing in most cases to size factor as main driving force of such variation (See (Angelini, Nobili, and Picillo (2011); Gabrieli (2011))), but finding great variation also among the large banks great level of dispersion (Iori, Gabbi, Germano, Hatzopoulos, Kapar, and Politi (2014)). Our results suggest instead another interpretation of such results that can be reconciled with the tiered structure of the market and the exploitation of market power by the core banks in the network.

1.4.2 Relevance of banks' market positioning

Informed on both the literature on liquidity hoarding and empirical findings, we want to shed light on the functioning of the e-MID interbank market in the aftermath of the major stock to the banking industry, namely the bankruptcy of Lehman Brothers, our first working hypothesis:

WORKING HP 1. Bank market positioning significantly affect markups and transaction amounts in times of market distress. Specifically, core banks were able to hoard liquidity through borrowing at lower cost and lending at higher prices.

Proceeding with formal parametric tests, our objective is to document which banks were

able to access the market after the onset of the Lehman crisis and what terms. One view is that shock led to a market-wide collapse of the e-MID market and prevented even banks that are good credit risks from accessing to the market. Accordingly we look at different dimensions of access to credit such as the interest rate at which banks borrow, the amount of the loan and the dispersion around market weighted interest rate. The last two are particular relevant since many participants in the interbank e-MID market suggest that the credit risk is managed via credit rationing rather than interest rates. To this aim, we allow market conditions to vary in different time windows around the bankruptcy of Lehman Brothers and indicate with binary variables the following periods:

- September 16, 2008 to October 16, 2008 (1m_postlehdman)
- October 17, 2008 to November 16, 2008 (2m_postlehdman)
- November 17, 2008 to December 16, 2008 (3m_postlehdman)
- December 17, 2008 to March 16, 2009 (2q_postlehdman)
- March 17, 2008 to June 16, 2009 (3q_postlehdman)
- June 17, 2008 to September 16, 2009 (4q_postlehdman)

In the regression we indicate with a binary variable the time windows. We analyze four main variables to assess conditions in the e-MID market: access, price, amount and spread relative to weighted market average. Price of the transactions is calculated as the weighted average of bank pair transactions in a given day and it is expressed as annual rate. The amount of the e-MID loans is expressed in millions of Euros. In our analysis, we study the intensive margin of credit, by calculating the daily volume weighted average interbank interest rate for each borrower and each lender at day t . Extensive margin of credit is studied by considering whether a bank had accessed the market. Intensive margin is further studied by considering the spread of granted loans, defined as the difference between loans rate and market interest rate. We include in the analysis only banks that were observed borrowing or lending in the market in the three data subsamples ⁹.

⁹We obtain similar results including also banks that were not active consistently in the market during the three phases. We chose however this subsample to limit possible confounding effects given by survival bias.

Our base model estimation is;

$$Y_{k,t} = \beta(\textit{Time Window}) + \mathbb{1}\{\textit{Periphery}\}_{k,\bar{q}} + \delta(\textit{Time Window} \times \mathbb{1}\{\textit{Periphery}\}_{k,t=\bar{q}}) + \epsilon_{k,t} \quad (1)$$

where k indexes banks borrowers (lenders), t indexes time in days, $\mathbb{1}\{\textit{Periphery}\}_{k,t=\bar{q}}$ is an indicator variable indicating the membership of the bank the periphery in the quarter before lehman crisis (\bar{q})¹⁰. Our dependent variables of interest $Y_{k,t}$ are $\textit{Access}_{k,t}$ equal to one when a bank borrows (lends) on a given day, $\textit{Amount}_{k,t}$ is the (log) amount borrowed (lent) in the e-MID market in a given day, $\textit{Rate}_{k,t}$ weighted price of the transaction a bank borrowed (lent) money in a given day, $\textit{Spread}_{k,t}$ the difference between $\textit{Rate}_{k,t}$ and weighted average market interest rate paid during the same day.

For the analysis in Tables 6 to 12 we aggregate the data in two sample so that we can examine the relevance of borrowers and lender positioning in the market. In the first sample, we aggregate all the e-MID overnight transactions to each borrower occurred in a given day, which consists of 40.322 observations. In the second sample instead we aggregate all the overnight transaction for each lender occurred in a given day for a total of 60.152 observations. Each one of these analysis is conducted with one observation per borrower-day (lender-day). In Table 6, we first look at the effect of Lehman’s bankruptcy on the e-MID interbank market. We do not control for fixed bank effects in this regression to verify first the aggregate impact of the crisis in our dependent variables. The first column is a probit estimation with a dependent variable equal to one if a bank borrows on a given day with dummy variables for the intermediate phase between two Bear Stearns collapse and the period after Lehman Bankruptcy. In columns (2), we compare the results for the chosen time windows including time dummies for the months and quarters immediately after Lehman collapse. Column (3)-(4) report the effect on the logarithm of the amount borrowed and

¹⁰Given the persistence of the core composition as shown in Section ??, we opt for a static definition of core and periphery and use the indicator variable for periphery and core membership identified with Craig and Von Peter (2014) algorithm in in the quarter before Lehman crisis. Results are robust to use indicator variables measured on previous two quarters.

columns (5)-(6) the volume weighted price for a given borrower. Results show consistently market access decreased immediately after the shock in the market. The abrupt cessation of activity is in line with the theoretical predictions by both counterparty and liquidity hoarding theories. We estimate negative coefficients becoming significantly more negative from the third month after Lehman's bankruptcy. Similarly also amounts of transactions declined significantly particularly if we consider the aftermath of Lehman bankruptcy.

Interest rate charged by banks first increase after the two Bearn Stearns as illustrated in the Figure 9 but then there is a significant negative shift after the Lehman bankruptcy, with average loan size of banks borrowing showing a coefficient estimate of $-.56$ in the aftermath of Lehman bankruptcy, which translates into a reduction in borrowing of nearly 40%. Running the same regression with fixed effects, the coefficient for the negative interest rate remain negative and significant. This suggests that after Lehman's bankruptcy the average rate applied by banks in the market decreased, but also that there has been an important redistribution of rates across different borrowers¹¹. The mixed results on interest rates offer first evidence that counterparty risk theories are insufficient in explaining the outcome of the crisis. Since the Lehman failure was widely considered the main trigger to worries over system wide counterparty risk, a decrease in interest rate is inconsistent with the jump in credit risk.

In Tables 7 and 8 we begin to disentangle the impact of the Lehman collapse on banks in different parts of the network. If lenders respond to the crisis by hoarding liquidity, we would expect to find an aggregate decrease in the amounts borrowed by the periphery as core banks will lend less as well as a decrease in the amount borrowed by the core from the core. Hence, we next add the interaction of bank membership in the periphery with time period dummies to the specification. The end result is a difference-in-difference estimation aimed to evaluate whether the market becomes more sensitive to the banks' market positioning after the 15 September 2008. Column (2) of Table 7 shows that borrowed amount of the periphery decreased immediately after Lehman collapse, in particular at the fourth quarter

¹¹Results available upon request from the author.

after the Lehman collapse. Column (4) and (6) show instead that the higher price and the spread by periphery borrowers indicate that for these banks that could still access the market obtaining liquidity was more costly with respect core banks in the market. This results serves as preliminary evidence for the presence of liquidity hoarding varying with network induced market power - core banks had the market power to obtain more funding and at lower cost than their periphery counterparts when borrowing from core banks. At the same time, this result goes against the theory of counterparty risk surges. Since the direct exposure of the financial crisis was concentrated on the balance sheet of internationally active banks mainly in the core, a lower price charged to them versus periphery banks is out of line with credit risk responses.

A similar trend is shown in Table 8 in which the average amount of loans lent by periphery decreased immediately after the crisis. Results are qualitatively similar for the different time windows under study. And lending decreases particularly for the last quarter of the sample. The impact on prices is economically significant with a negative coefficient for the second month after the Lehman collapse estimated at -0.35, which implies an interest rate applied to transactions with periphery banks as lender on average 29% less than core banks in the network. It is worth emphasizing that also spread for lenders in the periphery yields a statistically significant negative estimate for all the time windows under examination. This further corroborates our result that the ability to hoard and bargain over the price of liquidity is varying positively with market power and hence negatively with being a periphery lender.

Nevertheless, one may worry that the distinction by core and periphery banks is still correlated with direct balance sheet exposures, potentially confounding the previous results. Hence, we split the sample of banks in two group by bank balance sheet size, including “Major”, “Large” and “Foreign” banks in the same bin (“Large Banks”) and aggregate the rest of the bank population in a second group composed mainly of small and medium banks (“Small&Medium Banks”). This is because the former group were the ones active on and hence exposed to international capital markets before the Lehman failure. Verifying our results excluding this group offers further evidence against counterparty risk explanations

and support for the significance of market power in hoarding liquidity. If better data were available, more granular distinctions about credit risk could have been made. For example, if bank names were known, we could match them to Moody's ABCP exposure database to track down ABCP exposure from off-balance sheet vehicles that later return on the books. Nevertheless, looking at documentation about the nature of exposures versus bank type, we are fairly confident in our classification to rule out most endogeneity concerns.

Controlling for bank size, we first see that there exists some heterogeneity in the access to the market among borrowers in the periphery. Table 11 shows that large banks in the periphery were largely impaired in their access to the market (Column 4). We run the same specification then on the first group but excluding foreign banks (Column 5 and 6), to assess whether the results are driven by our inclusion in the group of international banks. As pointed out by Freixas and Holthausen (2004), asymmetric information problems between national and international banks might be heightened during the crisis, causing cross-border market segmentation and widening the price of the transactions differentials. In general, we do not find large differential between the previous estimates and those we run on the subsample of banks not including foreign banks. In contrast we find that the negative coefficient on large periphery borrowers becomes even larger¹². On the other hand, mixed on the coefficients for *Amount* are found in column (2) of Table 11, indicating that some banks in the group of small and medium banks were able to access the market. Similar results are shown considering the access of the market for the two groups of banks (Table 10).

To investigate further the mixed results in found using *Access* as dependent variable, we look at the daily amount borrowed by small and medium banks in the periphery. In line with the previous results, Table 11 shows that small and medium banks increased amount borrowed after the Lehman crisis (column 2), but despite the increase in the amount borrowed, the negative coefficient on the interaction term between our dummy indicator for periphery and *Rate* (Columns 3 and 4) as well as that for *Spread* (Columns 5 and 6) are

¹²Our results are in line with Angelini, Nobili, and Picillo (2011), which including controls for the nationality of the borrowers and lenders find very low values and typically not significant in the aftermath of Lehman collapse.

consistent with our theory predicting higher funding costs for small and medium borrowers in the periphery with respect to those in the core due to their lack in market power.

Running the same regression for lenders in the periphery and controlling for bank size (Table 12) we see that periphery lenders obtained a significantly lower return for their funding compared to core lenders, consistent with lower market power in the periphery ¹³.

1.4.3 Link between market structure and banks market power

In the previous section, we have demonstrated how advantage of core banks in their access to funding over periphery banks. However, other characteristics might differ between core and periphery banks. Hence, we further utilize finer variations within the core to verify that the main channel at play is indeed higher market power.

Specifically, since core banks are almost completely connected between themselves, one sufficient statistic for the measurement of market power within the core is the number of directly connected periphery lenders and borrowers i.e. the number of direct counterparty banks. This seemingly intuitive measure stems from solid microfoundations.

First, following the industrial organization literature, we can view the intermediary versus periphery lender and intermediary versus periphery borrower as two two sided markets. Hence, the degree of monopoly power and hence the mark up in price of the intermediary bank is given by the number of agents on the other side.

Alternatively, given the stickiness and persistence of links in the interbank market, we can also adopt the perspective of a bargaining game with renegotiable contracts. In this case, the outside option in bilateral bargaining between a given periphery and intermediary is the equilibrium outcome of the remaining network. Hence, the larger the number of lenders (borrowers) bargaining with an intermediary, the higher its outside options and hence the larger the split of surplus in borrowing (lending). We build our proxy of market power as

¹³To further check our interpretation we run also a regression using as unit of observation the pair of banks transacting in a given day. Controlling for links from the core to periphery we run the baseline specification with the time dummies on *Amount* and *Rate*. Results are consistent with our interpretation and suggest a significant decrease in the flows of liquidity from core to periphery and more advantageous terms for transaction between core lenders and borrowers in the periphery.

the interaction term between core indicator variable and the (log) number of counterparties of the banks in the quarter before the Lehman collapse.

With the theoretical foundations in mind, we proceed to test following working hypothesis:

WORKING HP 2. *The channel through which core banks are advantaged to obtain and provide liquidity at better terms is through improved market power.*

Our regression takes the following general form:

$$Y_{ij,t} = \beta(\text{Time Window}) + \mathbb{1}\{\text{Core}\}_{j,\bar{q}} \times \log \text{bor count}_{j,\bar{q}} + \delta(\text{Time Window} \times \mathbb{1}\{\text{Core}\}_{j,t=\bar{q}} \times \log \text{bor count}_{j,\bar{q}}) + \epsilon_{ij,t} \quad (2)$$

where $Y_{ij,t}$, indicates our pair level dependent variable for a given transaction between lender i to borrower j in a given day t ; $\log \text{bor count}_{j,\bar{q}}$ our preferred measure of market power; and $\epsilon_{ij,t}$ the error term.

For Tables 14 to 19 we use as unit of observation in the analysis the pair of banks transacting in the interbank in a given day. To be consistent with our previous analysis, we aggregate thus multiple loans on the same day for the same bank pair to one observation and compute a volume-weighted average interest rate and spread at the bank pair level.

We aim to capture how banks in the core with relatively larger number of counterparties are able to manipulate prices and limit the access liquidity for banks with less market power, that are in the periphery of the network. A valid test requires that variation in the dependent variable is independent of banks' lending opportunities. We address this identification problem, by using a within-bank estimator which allows us to identify the impact of bank market power.

The use of the within-estimator imposes us to restrict the set of observation to only those banks that borrow (lend) from more than one bank in a given day. This allow us to compare the supply of liquidity across banks in the same tier of the interbank network

for a given lender. By comparing loan term across counterparties of the same bank, we control for bank's lending opportunities and identify the effect of bank market positioning on bank lending behaviour and market power. Standard errors are clustered on the lending bank. I expect that banks that borrow from banks with higher market power have a larger differential in the price of the transactions and that the latter consistently absorb liquidity.

Table 15 examines formally the variation in the terms of loans for links from the periphery to the core. We run the regressions on *Amount* and *Rate* variables. Results show that in the first three months in the aftermath of the collapse Lehman Brothers amount borrowed by banks in the core with higher market power increased significantly (Column 1). The effect remains economically significant when we introduce also our within-estimator (Column 2) for almost all the time windows under consideration. As an example, the positive coefficient in the second quarter after 15 Sep 2008, 1% increase in the number of counterparties translates in almost 1% higher capacity of borrowing liquidity from banks in our treatment group. Columns (3) and (4) report the capacity of such banks also to absorb liquidity at less onerous terms consistently in the quarters and months after the crisis.

Table 14 supports our previous findings. Here we control the sample observations for links from the periphery to the core and analyse as before trading behaviours of banks joining higher level of market power through their network positioning. Column (2) and Column (4) show that core borrowers with higher number of connections can exercise their market power also within the core. Comparing banks in the core. The economic significance of the coefficient remains stable in the time windows considered after Lehman bankruptcy considered with a negative coefficient around -0.3.

The signs on the coefficients of the indicator variables are consistent with prior findings in the literature and economic relevance. They are also consistent with our view that market power is closely related to bank positioning in the network.

1.4.4 Robustness

We already mentioned some of our robustness tests. To simplify our representation in discussing these, we will say that results are “qualitatively similar” only when the sign and significance of the coefficient on the cross-term between our core (periphery) indicator variable and market power proxy remain unchanged for the time windows we consider in the analysis. We will note explicitly the cases in which the sign of the coefficient changes or its level of significance declines (All of the results here are available from the author).

Our results are robust to the use the definition of core and periphery membership for different quarters before the financial crisis. We tested using the first quarter of the sample the core definition resulted from running the algorithm of Craig and Von Peter (2014) two quarters before Lehman failure.

We also reestimated the parametric tests on the link between market structure and banks lending behaviour replacing our preferred measure of market power with the ranking of banks for different centrality measures. We tried in particular betweenness centrality, eigenvector centrality and coreness calculated as proposed in Della Rossa, Dercole, and Piccardi (2013).

We run also a battery of robustness checks to verify whether our results in bank pair regression are robust to potential counterparty risk coundfunding effects. First, we rerun Eq. 2 only on the subsample of banks less likely to be affected by counterparty risk (Tables 16 and 17) as explained in the previous Section. Second, in Tables 18 and 19 we perform the same regression on only Italian banks. We show that results remain qualitatively unchanged.

1.5 Conclusions

We have shown that the Italian overnight interbank network displays a core periphery structure. We highlight the economic significance of this tired structure utilizing the shock to interbank markets in the recent financial crisis. Specifically, being in the core means having lower cost of funding while enjoying a higher benefit from lending especially in periods of market distress. We further evaluate the mechanism at play and demonstrate that market power, and equivalently bargaining power, was at play in driving the wedge between the two

groups of banks.

We have also shown how competing theories proposed in the literature are inconsistent with empirical observations. Counterparty risk explanations do not seem of first importance since our results remain robust upon exclusion of exposed large and foreign banks. Further, we carried out our analysis with the implicit assumption of complete information. We believe that this simplification is a realistic one because given the nature of the recent crisis, asymmetric information should have been more worrisome for borrowing banks with assets suffering from increased information sensitivity during the crisis. These were arguable big and internationally exposed banks in the core, for which we witnessed precisely the opposite - a decrease in borrowing rates.

Nevertheless, our data was not perfect and more detailed analysis should be conducted as more information becomes available. For example, knowledge about maturity structure of bank assets and liabilities would allow us to test how the riskiness of liquidity shortage interacts with better bargaining power. Further, with information about banks' capital market financing, we could look at how access to money markets and bond markets related to interbank market power and network position. Do they reinforce or balance each other out? Finally, given the substitution between ECB liquidity facilities and interbank markets, to what extent can central bank policy smooth the effects of heterogenous market power on interbank loan prices?

At this point, our findings from the given data already yield important contributions. We address the puzzle on liquidity hoarding in the literature by showing that it exists but to a different extent depending on the network position induced bargaining power of banks.

Policy wise, our results offers invaluable information for liquidity measures during crisis events. In the presence of persistent core periphery tiered structure between financial institutions, a consistent supply of liquidity by a central bank may not necessarily "cross "the system due to the presence of the key players extracting contractual surplus from the banks in the periphery with less market power. Hence, in order to ensure the smooth functioning of the banking system, any measure should ensure that periphery banks, which account for

the majority of banks and lending to the real economy, are not precluded.

Even in normal times, conventional monetary policy ought to take the core periphery structure seriously. While we utilized the crisis as a laboratory to understand an amplified event, bargaining power is always present in the market. For example, when central bank rates increase, the cost of outside options for borrowing from the core increases, and core banks should be able to make use of their higher bargaining power to obtain a higher mark up. This would suggest that a relationship between monetary policy rates and financial stability via the channel of market power in the interbank market, as pointed out by Duffie and Krishnamurthy (2016).

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Figures

Figure 1: Daily Transaction Amounts

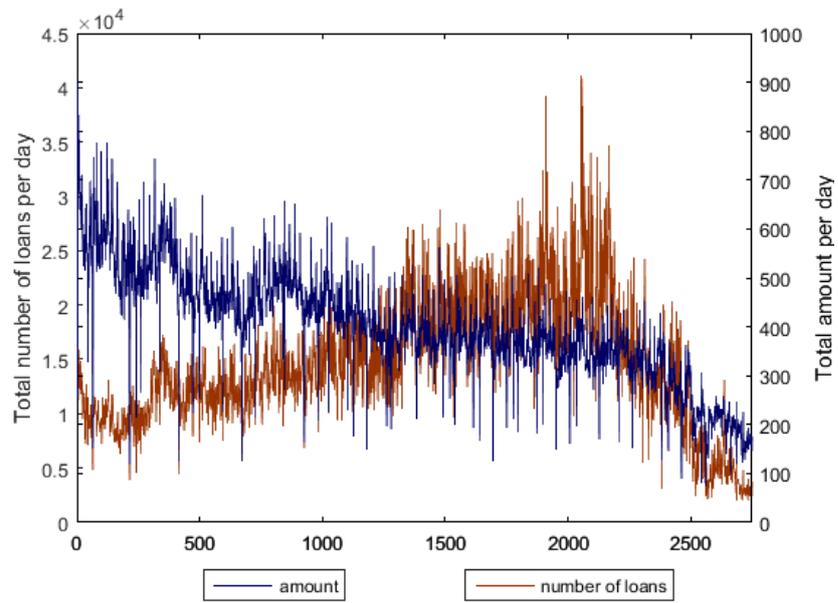


Figure 2: Daily Number of Borrowers and Lenders

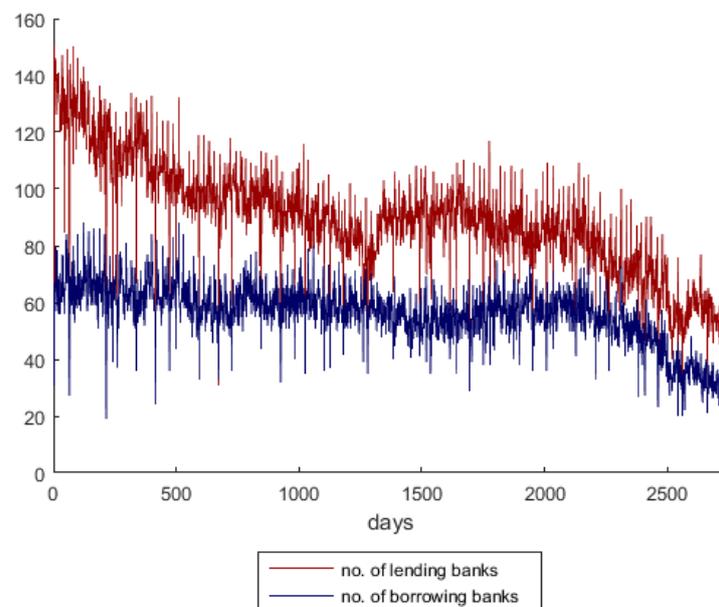


Figure 3: Network Centralization - QII 2006

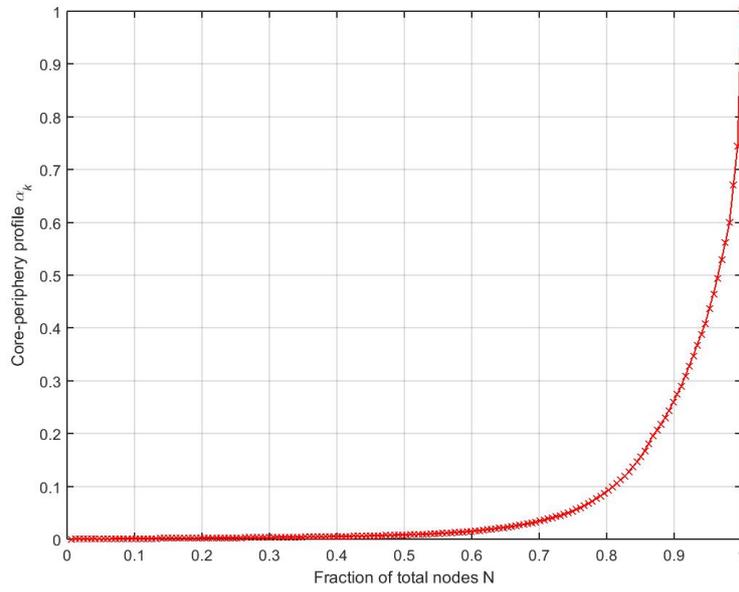


Figure 4: Network Centralization - QIII 2009

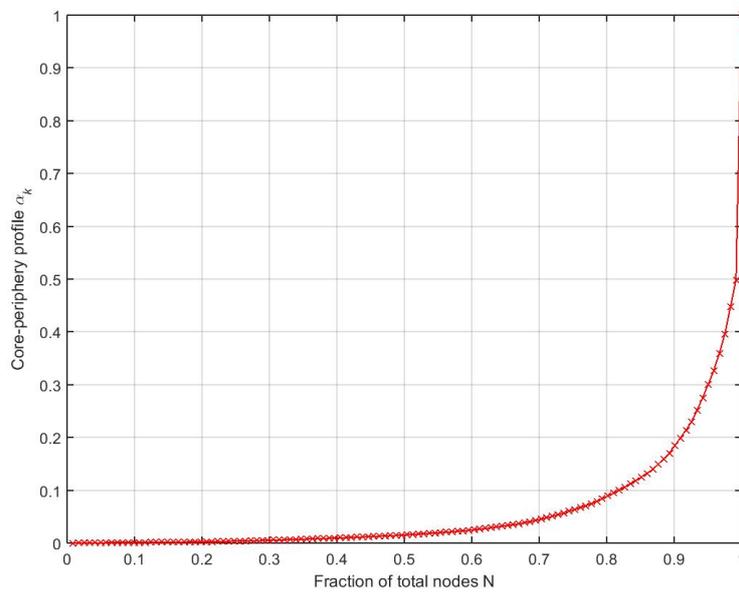


Figure 5: (Mean) Daily Transaction Amounts

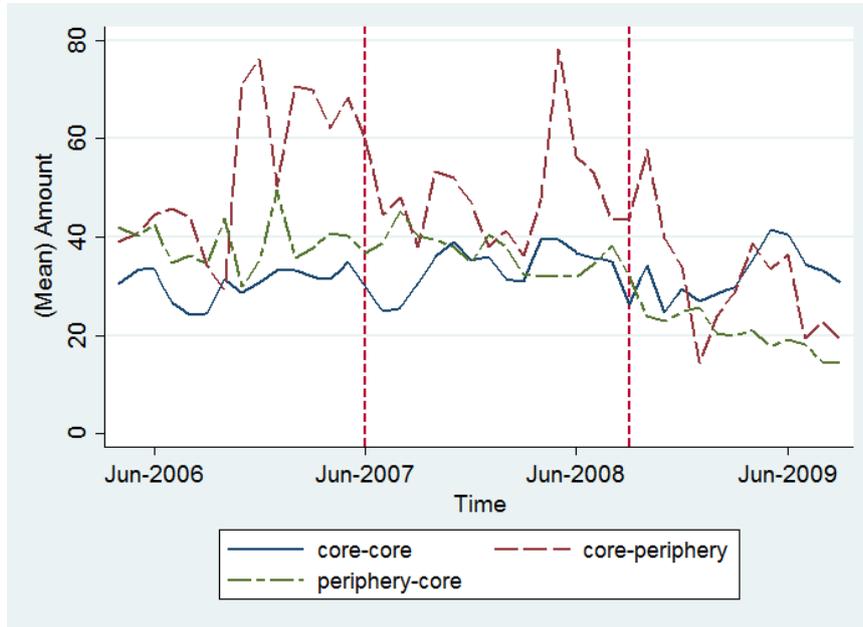


Figure 6: (Median) Daily Transaction Amounts

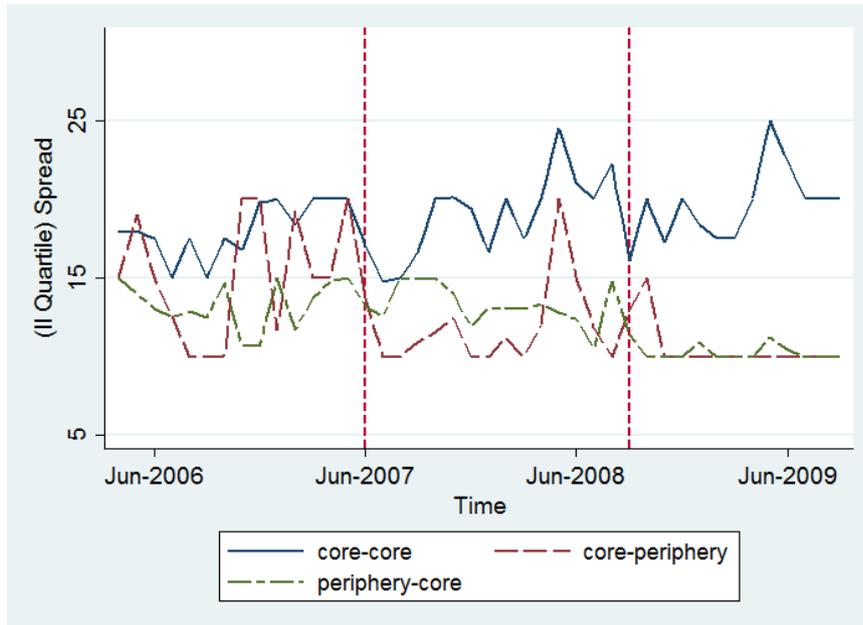


Figure 7: (Upper Quartile) Daily Transaction Amounts

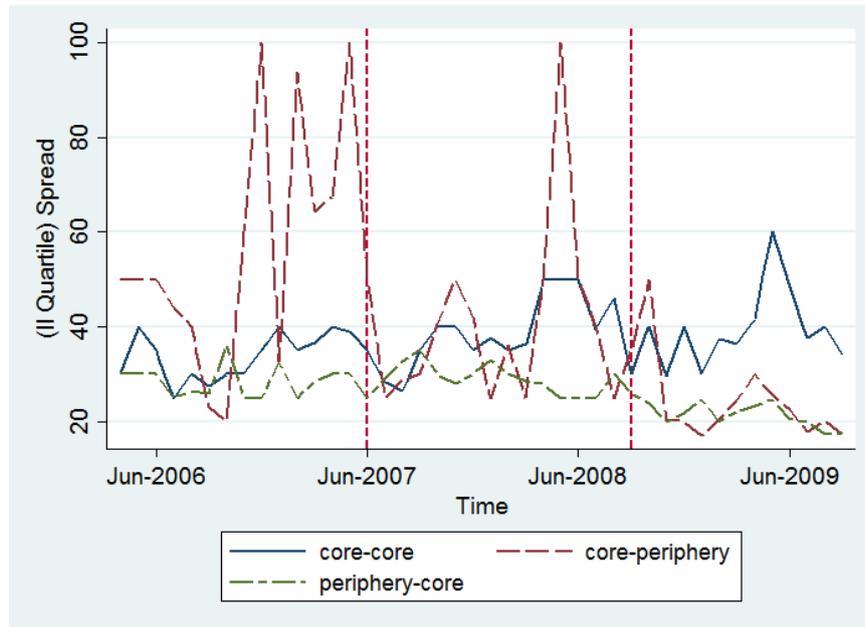


Figure 8: (Lower Quartile) Daily Transaction Amounts

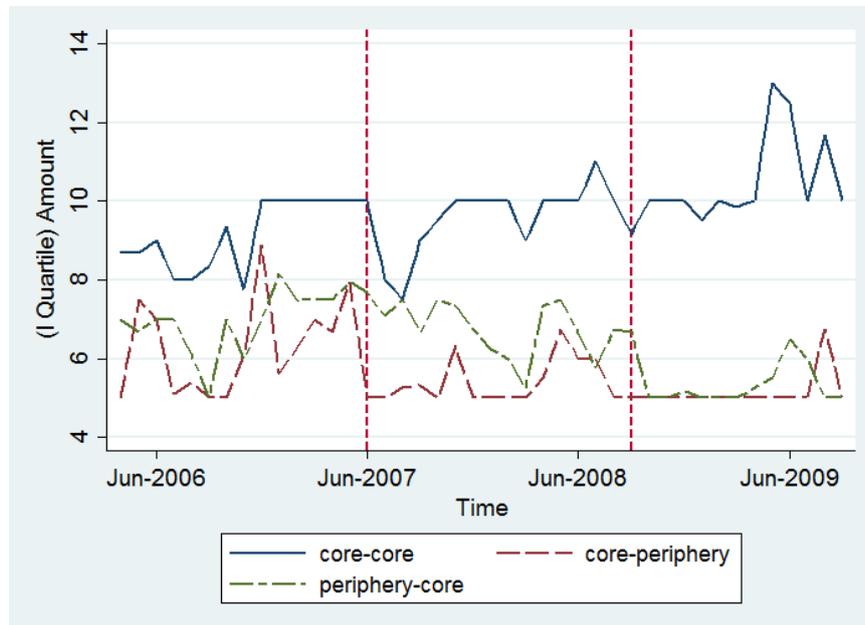


Figure 9: (Mean) Weighted Transaction Rate

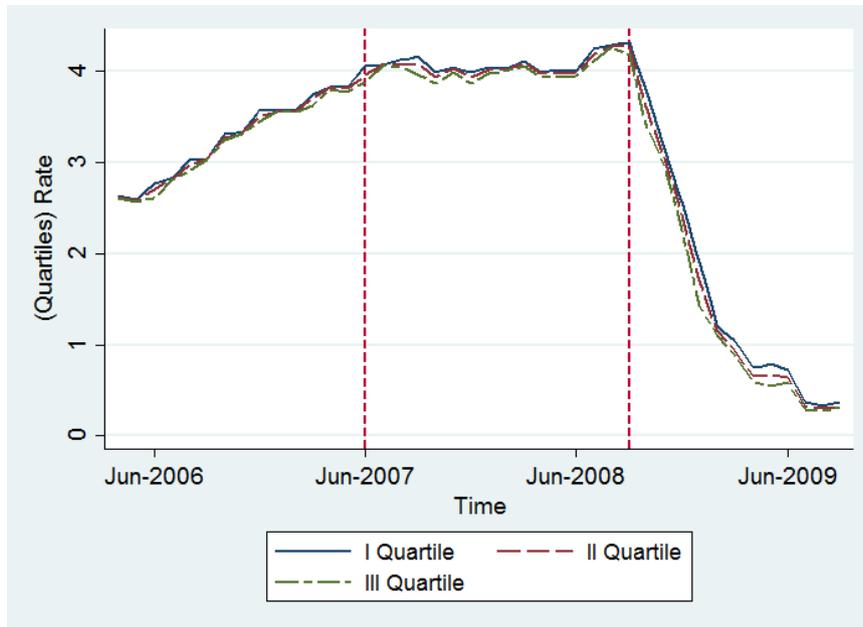


Figure 10: (Mean) Weighted Transaction Spread

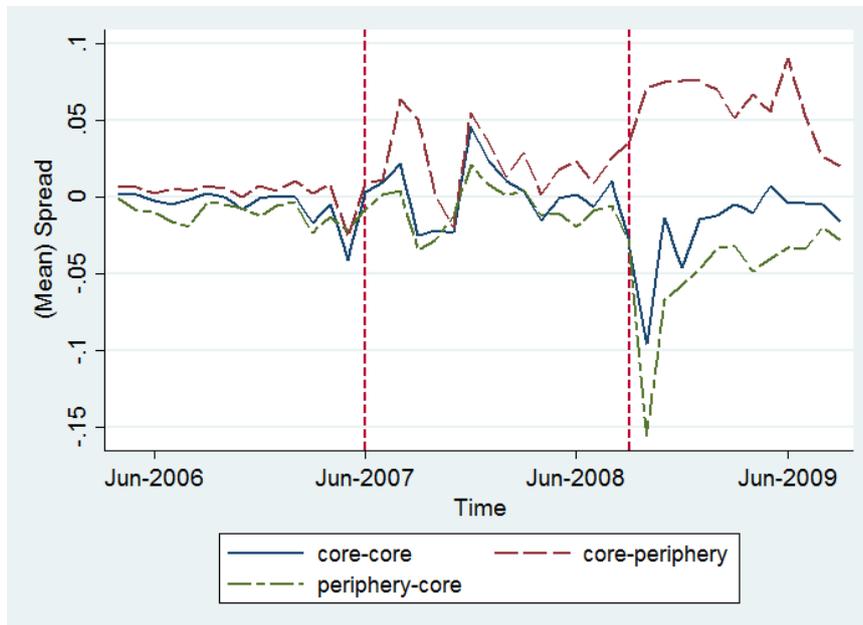


Figure 11: (Median) Weighted Transaction Spread

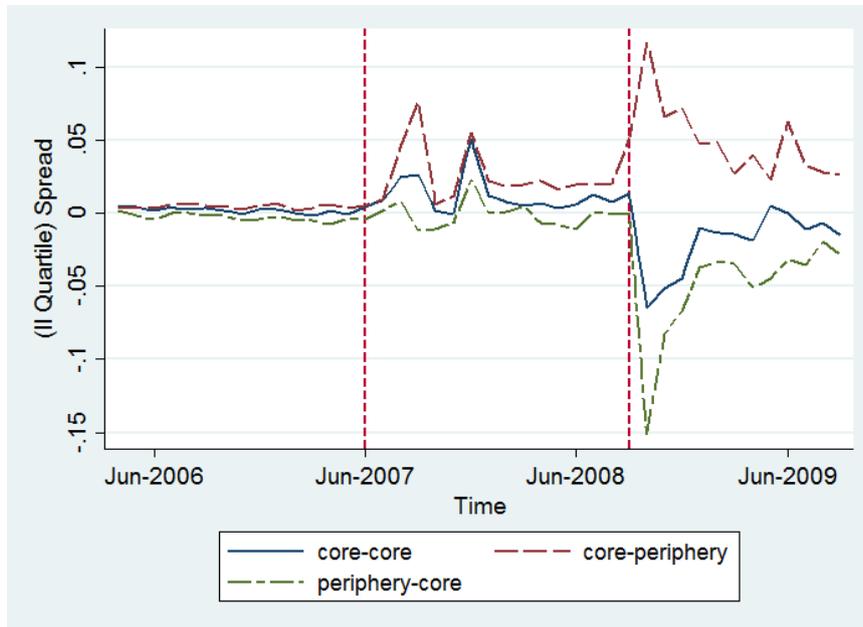


Figure 12: (Lower Quartile) Weighted Transaction Spread

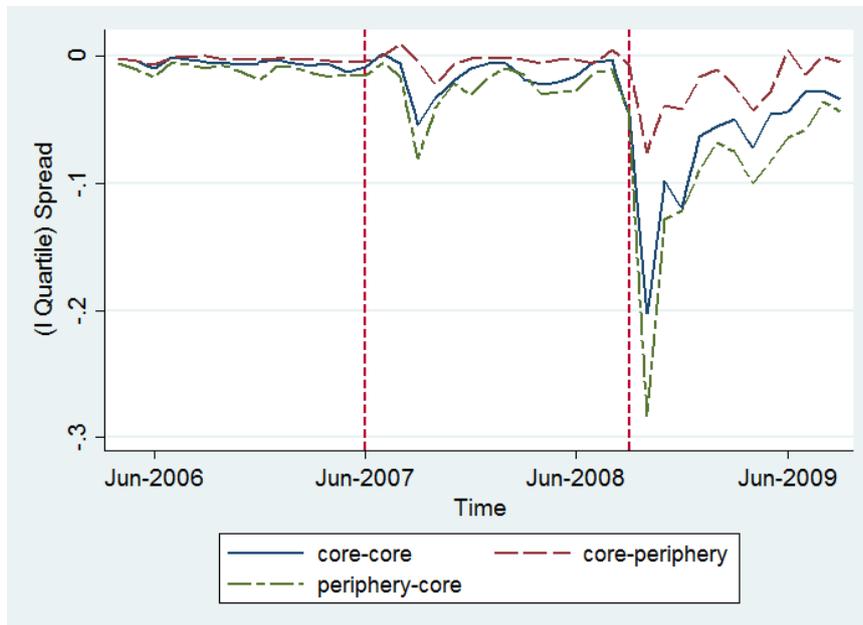
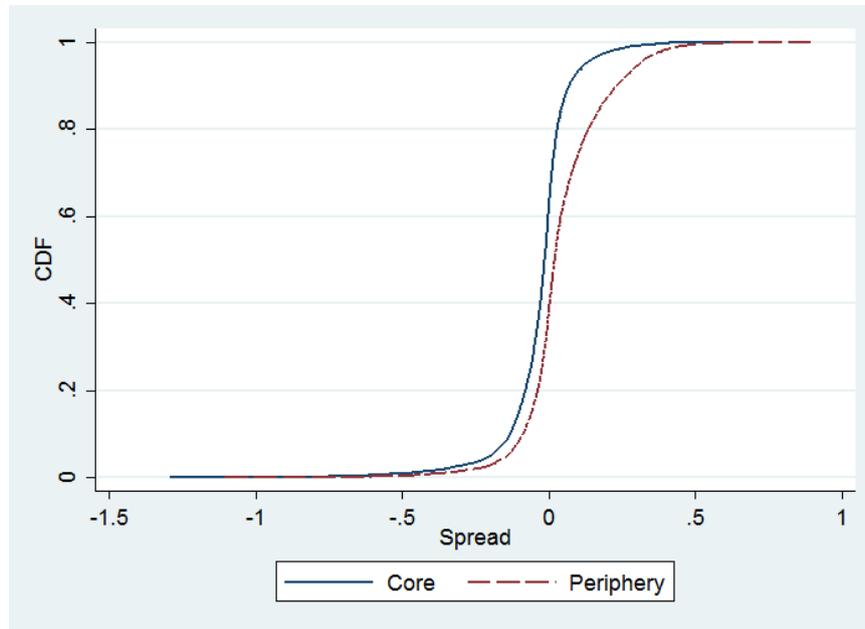


Figure 13: Transaction Spread - Cumulative Distribution (Jul-08/Sep-09)



Tables

tab:cores summary statistics

Table 1: Core-Periphery Profile by Quarter

Quarter	N. Banks	N. Cores	Fraction of Cores	CP-centralization
200603	165	31	0.19	0.89
200606	167	34	0.20	0.86
200609	168	32	0.19	0.88
200612	170	34	0.20	0.85
200703	167	34	0.20	0.80
200706	167	35	0.21	0.78
200709	170	34	0.20	0.83
200712	167	32	0.19	0.77
200803	166	30	0.18	0.77
200806	166	29	0.17	0.82
200809	161	28	0.17	0.87
200812	144	24	0.17	0.87
200903	142	21	0.15	0.82
200906	144	21	0.15	0.95
200909	121	23	0.19	0.88

Table 2: Transition Matrix

	Core	Periphery	Out
Core	0.80	0.19	0.01
Periphery	0.04	0.90	0.06
Out	0.00	0.03	0.97

Table 3: Lending Banks Size Distribution

Banks Size	2006	2007	2008	2009
MA	6	6	4	4
GR	9	9	6	4
ME	15	14	12	10
PI	59	56	54	54
MI	17	19	21	22
FB	65	70	70	43

Table 4: Borrowing Banks Size Distribution

Banks Size	2006	2007	2008	2009
MA	6	6	4	4
GR	9	9	6	5
ME	13	12	12	10
PI	50	54	53	50
MI	14	16	20	16
FB	58	62	58	37

Table 5: Summary Statistics of Transaction Variables

Transaction Variables	mean	p50	p5	p95	min	max
spread len	.0019987	.0004569	-.0792687	.0926056	-1.184701	.896391
spread pair	.0028637	.0011555	-.1033285	.1282351	-1.498211	.896391
rate bor	3.208687	3.592605	.4731463	4.271634	.15	4.9
rate len	.7195687	.3870968	.0341176	3	.0000454	5
rate pair	3.203634	3.585	.45	4.27	.15	5.033333
amount pair	50.23409	15	3.48	200	.05	4000
amount bor	492.8776	275	21	1709.82	.05	12245
amount len	312.1521	130	15	1330	.05	9420
amount foreign len	105.4591	0	0	688	0	9150
amount foreign bor	232.4732	93.9	0	1048.2	0	4744.5

Table 6: Dependent Variables, Borrowers

	Access		Amount		Rate	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman		-0.209*** (-8.96)		-0.190*** (-4.06)		0.372*** (20.08)
m2_postlehman		-0.306*** (-12.28)		-0.353*** (-6.79)		-0.291*** (-14.19)
m3_postlehman		-0.273*** (-11.27)		-0.277*** (-5.61)		-0.833*** (-42.66)
q2_postlehman		-0.311*** (-20.73)		-0.428*** (-13.70)		-2.193*** (-177.82)
q3_postlehman		-0.353*** (-23.65)		-0.385*** (-12.20)		-2.928*** (-234.86)
q4_postlehman		-0.434*** (-29.05)		-0.638*** (-19.74)		-3.248*** (-254.67)
post_stearns	-0.084*** (-10.06)		-0.224*** (-14.16)		0.756*** (108.21)	
post_lehman	-0.396*** (-43.08)		-0.568*** (-30.31)		-1.896*** (-229.74)	
N	130666	130666	40322	40322	40322	40322

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 7: Dependent Variables, Borrowers

	Amount		Rate		Spread	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	-0.156*** (-3.36)	-0.021 (-0.30)	0.371*** (20.03)	0.283*** (10.14)	-0.009*** (-3.24)	-0.070*** (-17.44)
m2_postlehman	-0.335*** (-6.53)	-0.210*** (-2.61)	-0.291*** (-14.21)	-0.376*** (-11.68)	-0.027*** (-9.09)	-0.086*** (-18.56)
m3_postlehman	-0.242*** (-4.96)	-0.074 (-1.01)	-0.833*** (-42.70)	-0.916*** (-31.23)	0.021*** (7.24)	-0.025*** (-6.02)
q2_postlehman	-0.401*** (-13.00)	-0.107** (-2.26)	-2.194*** (-177.82)	-2.226*** (-117.14)	-0.010*** (-5.66)	-0.037*** (-13.73)
q3_postlehman	-0.353*** (-11.31)	-0.158*** (-3.34)	-2.929*** (-234.81)	-2.981*** (-157.27)	0.002 (1.31)	-0.030*** (-11.03)
q4_postlehman	-0.568*** (-17.77)	-0.237*** (-5.29)	-3.250*** (-254.25)	-3.299*** (-184.03)	0.002 (1.15)	-0.019*** (-7.51)
bper_precrisis2008	0.462*** (32.28)	0.560*** (34.37)	-0.012** (-2.14)	-0.035*** (-5.37)	0.023*** (27.69)	0.009*** (9.59)
m1_postlehmanXperiphery_borrower		-0.228** (-2.45)		0.154*** (4.14)		0.108*** (20.12)
m2_postlehmanXperiphery_borrower		-0.204* (-1.96)		0.141*** (3.38)		0.098*** (16.32)
m3_postlehmanXperiphery_borrower		-0.289*** (-2.95)		0.146*** (3.72)		0.081*** (14.31)
q2_postlehmanXperiphery_borrower		-0.502*** (-8.05)		0.053** (2.14)		0.046*** (12.84)
q3_postlehmanXperiphery_borrower		-0.334*** (-5.31)		0.090*** (3.56)		0.056*** (15.44)
q4_postlehmanXperiphery_borrower		-0.657*** (-10.30)		0.095*** (3.73)		0.040*** (10.93)
N	40322	40322	40322	40322	40322	40322

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 8: Dependent Variables, Lenders

	Amount		Rate		Spread	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	-0.181*** (-5.05)	0.158** (2.09)	-0.016 (-0.48)	0.162** (2.32)	-0.012*** (-5.55)	0.050*** (10.61)
m2_postlehman	-0.287*** (-6.96)	0.042 (0.46)	-0.069* (-1.82)	0.210** (2.51)	-0.046*** (-18.10)	0.007 (1.18)
m3_postlehman	-0.260*** (-6.69)	0.076 (0.88)	-0.290*** (-8.11)	-0.107 (-1.34)	0.015*** (6.21)	0.065*** (12.14)
q2_postlehman	-0.437*** (-18.34)	-0.217*** (-4.22)	-0.893*** (-40.71)	-0.786*** (-16.57)	0.020*** (13.58)	0.068*** (21.39)
q3_postlehman	-0.436*** (-18.73)	-0.001 (-0.02)	-1.254*** (-58.48)	-1.127*** (-23.30)	0.028*** (19.50)	0.068*** (20.87)
q4_postlehman	-0.604*** (-24.65)	-0.129** (-2.24)	-1.408*** (-62.44)	-1.246*** (-23.48)	0.012*** (8.19)	0.039*** (11.07)
lper_precrisis2008	0.112*** (7.95)	0.223*** (13.75)	0.130*** (9.99)	0.177*** (11.81)	-0.027*** (-30.64)	-0.013*** (-13.27)
m1_postlehmanXperiphery_lender		-0.432*** (-5.02)		-0.227*** (-2.86)		-0.079*** (-14.93)
m2_postlehmanXperiphery_lender		-0.412*** (-4.05)		-0.350*** (-3.74)		-0.067*** (-10.57)
m3_postlehmanXperiphery_lender		-0.419*** (-4.31)		-0.229** (-2.55)		-0.063*** (-10.41)
q2_postlehmanXperiphery_lender		-0.276*** (-4.75)		-0.134** (-2.51)		-0.061*** (-16.90)
q3_postlehmanXperiphery_lender		-0.540*** (-9.23)		-0.157*** (-2.91)		-0.049*** (-13.54)
q4_postlehmanXperiphery_lender		-0.580*** (-9.13)		-0.198*** (-3.38)		-0.033*** (-8.38)
N	60152	60152	60152	60152	60152	60152

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 9: Impact of the Crisis on Banks Access, Large Vs Small&Medium Periphery Borrowers

	Small&Medium		Large		Large(IT)	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	-0.144*** (-4.62)	-0.004 (-0.06)	-0.403*** (-10.12)	0.008 (0.08)	-0.352*** (-4.19)	0.095 (0.78)
m2_postlehman	-0.190*** (-5.76)	-0.361*** (-5.92)	-0.619*** (-13.92)	-0.184* (-1.67)	-0.789*** (-9.02)	-0.233** (-1.98)
m3_postlehman	-0.117*** (-3.64)	-0.143** (-2.35)	-0.649*** (-14.69)	-0.067 (-0.60)	-0.710*** (-8.40)	-0.149 (-1.27)
q2_postlehman	-0.199*** (-9.99)	-0.314*** (-8.49)	-0.625*** (-23.31)	-0.104 (-1.54)	-0.611*** (-11.93)	-0.132* (-1.81)
q3_postlehman	-0.309*** (-15.45)	-0.326*** (-8.97)	-0.554*** (-21.65)	-0.184*** (-2.79)	-0.425*** (-8.61)	-0.016 (-0.21)
q4_postlehman	-0.263*** (-13.43)	-0.132*** (-3.68)	-0.996*** (-32.61)	-0.382*** (-6.09)	-0.684*** (-13.63)	-0.089 (-1.26)
periphery_borrower	-0.785*** (-70.94)	-0.786*** (-59.87)	-1.491*** (-80.63)	-1.317*** (-60.90)	-1.096*** (-44.36)	-0.845*** (-29.97)
m1_postlehmanXperiphery_borrower		-0.195*** (-2.77)		-0.469*** (-3.97)		-0.883*** (-4.92)
m2_postlehmanXperiphery_borrower		0.242*** (3.35)		-0.511*** (-4.22)		-1.433*** (-6.20)
m3_postlehmanXperiphery_borrower		0.036 (0.51)		-0.692*** (-5.70)		-1.319*** (-6.41)
q2_postlehmanXperiphery_borrower		0.163*** (3.71)		-0.614*** (-8.28)		-0.987*** (-8.86)
q3_postlehmanXperiphery_borrower		0.025 (0.57)		-0.429*** (-5.99)		-0.753*** (-7.31)
q4_postlehmanXperiphery_borrower		-0.187*** (-4.35)		-0.805*** (-10.91)		-1.399*** (-11.42)
N	70630	70630	60039	60039	12310	12310

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 10: Impact of the Crisis on Banks Access, Large Vs Small&Medium Periphery Lenders

	Small&Medium		Large		Large(IT)	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	0.110*** (3.20)	0.454*** (5.42)	-0.403*** (-13.41)	-0.072 (-1.12)	-0.139*** (-2.92)	-0.045 (-0.69)
m2_postlehman	-0.183*** (-5.38)	0.144* (1.84)	-0.597*** (-17.74)	-0.452*** (-6.13)	-0.299*** (-5.56)	-0.416*** (-5.49)
m3_postlehman	-0.172*** (-5.14)	-0.163** (-2.11)	-0.460*** (-14.40)	-0.092 (-1.31)	-0.132** (-2.51)	-0.035 (-0.48)
q2_postlehman	-0.074*** (-3.53)	-0.014 (-0.30)	-0.550*** (-27.44)	-0.085* (-1.95)	-0.160*** (-4.95)	-0.025 (-0.56)
q3_postlehman	0.007 (0.34)	-0.170*** (-3.53)	-0.582*** (-29.35)	-0.084** (-1.97)	-0.199*** (-6.27)	-0.025 (-0.58)
q4_postlehman	-0.104*** (-5.17)	-0.314*** (-6.55)	-0.817*** (-39.37)	-0.398*** (-9.39)	-0.392*** (-12.61)	-0.359*** (-8.24)
periphery_lender	0.494*** (32.35)	0.481*** (24.67)	-0.319*** (-28.87)	-0.200*** (-15.90)	0.447*** (31.52)	0.488*** (30.36)
m1_postlehmanXperiphery_lender		-0.417*** (-4.54)		-0.423*** (-5.82)		-0.193** (-2.03)
m2_postlehmanXperiphery_lender		-0.405*** (-4.67)		-0.182** (-2.20)		0.243** (2.23)
m3_postlehmanXperiphery_lender		-0.011 (-0.13)		-0.465*** (-5.90)		-0.202* (-1.92)
q2_postlehmanXperiphery_lender		-0.075 (-1.40)		-0.590*** (-12.02)		-0.277*** (-4.30)
q3_postlehmanXperiphery_lender		0.219*** (4.11)		-0.637*** (-13.20)		-0.356*** (-5.64)
q4_postlehmanXperiphery_lender		0.257*** (4.87)		-0.544*** (-11.14)		-0.065 (-1.04)
N	53497	53497	82178	82178	33085	33085

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 11: Impact of the Crisis on Dependent Variables, Small&Medium Periphery Borrowers

	Amount		Rate		Spread	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	-0.081** (-2.06)	-0.126** (-2.17)	0.353*** (15.34)	0.292*** (8.60)	-0.009*** (-3.00)	-0.062*** (-13.68)
m2_postlehman	0.013 (0.30)	-0.188*** (-2.73)	-0.330*** (-13.32)	-0.406*** (-10.07)	-0.055*** (-16.49)	-0.109*** (-20.35)
m3_postlehman	0.129*** (3.23)	-0.056 (-0.91)	-0.848*** (-36.44)	-0.916*** (-25.45)	0.019*** (5.93)	-0.019*** (-4.03)
q2_postlehman	-0.193*** (-7.52)	-0.150*** (-3.69)	-2.191*** (-145.91)	-2.253*** (-94.94)	-0.013*** (-6.53)	-0.045*** (-14.18)
q3_postlehman	-0.134*** (-5.01)	-0.070* (-1.73)	-2.947*** (-188.95)	-2.995*** (-126.87)	-0.001 (-0.46)	-0.035*** (-11.12)
q4_postlehman	-0.141*** (-5.59)	-0.099*** (-2.69)	-3.269*** (-221.36)	-3.309*** (-153.72)	-0.001 (-0.50)	-0.020*** (-6.94)
periphery_borrower	-0.105*** (-8.58)	-0.105*** (-7.49)	0.002 (0.29)	-0.021*** (-2.58)	0.027*** (28.16)	0.013*** (11.84)
m1_postlehmanXperiphery_borrower		0.084 (1.06)		0.111** (2.39)		0.096*** (15.63)
m2_postlehmanXperiphery_borrower		0.322*** (3.69)		0.123** (2.40)		0.087*** (12.86)
m3_postlehmanXperiphery_borrower		0.318*** (3.94)		0.116** (2.45)		0.065*** (10.36)
q2_postlehmanXperiphery_borrower		-0.072 (-1.38)		0.104*** (3.39)		0.053*** (12.95)
q3_postlehmanXperiphery_borrower		-0.113** (-2.11)		0.084*** (2.67)		0.060*** (14.28)
q4_postlehmanXperiphery_borrower		-0.080 (-1.59)		0.073** (2.47)		0.034*** (8.70)
N	26299	26299	26299	26299	26299	26299

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 12: Impact of the Crisis on Dependent Variables, Small&Medium Periphery Lenders

	Amount		Rate		Spred	
	(1)	(2)	(3)	(4)	(5)	(6)
m1_postlehman	-0.019 (-0.67)	0.111 (1.60)	-0.022 (-0.56)	0.489*** (5.16)	-0.031*** (-10.39)	0.022*** (3.05)
m2_postlehman	-0.133*** (-4.18)	-0.026 (-0.35)	-0.089** (-2.07)	0.306*** (3.08)	-0.078*** (-23.51)	-0.001 (-0.11)
m3_postlehman	-0.124*** (-4.02)	-0.001 (-0.01)	-0.306*** (-7.30)	0.104 (0.94)	0.008** (2.40)	0.058*** (6.90)
q2_postlehman	-0.057*** (-3.11)	0.211*** (4.41)	-0.773*** (-30.79)	-0.584*** (-8.96)	0.016*** (8.17)	0.116*** (23.47)
q3_postlehman	0.024 (1.38)	0.283*** (5.63)	-1.107*** (-45.82)	-1.017*** (-14.84)	0.024*** (12.81)	0.104*** (20.13)
q4_postlehman	0.017 (0.97)	0.466*** (8.86)	-1.237*** (-50.40)	-1.171*** (-16.33)	0.008*** (4.39)	0.051*** (9.40)
periphery_lender	-0.052*** (-3.48)	0.059*** (3.13)	-0.055*** (-2.69)	0.038 (1.46)	-0.038*** (-24.48)	-0.008*** (-3.87)
m1_postlehmanXperiphery_lender		-0.148* (-1.95)		-0.607*** (-5.85)		-0.061*** (-7.74)
m2_postlehmanXperiphery_lender		-0.120 (-1.48)		-0.480*** (-4.35)		-0.092*** (-10.99)
m3_postlehmanXperiphery_lender		-0.139 (-1.58)		-0.475*** (-3.97)		-0.057*** (-6.31)
q2_postlehmanXperiphery_lender		-0.311*** (-6.00)		-0.218*** (-3.09)		-0.116*** (-21.81)
q3_postlehmanXperiphery_lender		-0.293*** (-5.46)		-0.100 (-1.37)		-0.091*** (-16.52)
q4_postlehmanXperiphery_lender		-0.507*** (-9.05)		-0.073 (-0.96)		-0.048*** (-8.31)
N	30510	30510	30510	30510	30510	30510

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 13: Robustness II: Impact of the Crisis, Core → Periphery Links

	Amount	Rate
	(1)	(2)
m1_postlehman	0.067 (1.17)	0.520*** (20.90)
m2_postlehman	-0.131* (-1.68)	-0.200*** (-5.92)
m3_postlehman	-0.228*** (-3.41)	-0.701*** (-24.20)
q2_postlehman	-0.503*** (-12.71)	-2.077*** (-121.13)
q3_postlehman	-0.311*** (-7.33)	-2.773*** (-150.68)
q4_postlehman	-0.482*** (-9.65)	-3.126*** (-144.33)
N	21497	21497

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 14: Impact of the Crisis, Core → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-3.636*** (-4.62)	-4.063*** (-3.42)	1.776*** (3.78)	1.823*** (4.19)
m2_postlehman	-1.981 (-1.55)	-3.507** (-2.37)	0.131 (0.17)	0.561 (0.79)
m3_postlehman	-4.300*** (-4.70)	-5.160*** (-4.91)	0.250 (0.46)	0.545 (1.08)
q2_postlehman	-6.363*** (-11.03)	-6.679*** (-4.37)	-1.040*** (-3.02)	-0.698 (-1.55)
q3_postlehman	-9.871*** (-14.90)	-9.616*** (-4.16)	-1.865*** (-4.71)	-1.474*** (-4.05)
q4_postlehman	-4.266*** (-6.03)	-5.168*** (-3.44)	-2.576*** (-6.10)	-2.342*** (-6.69)
core borrowerXlog bor count	0.192*** (19.22)	0.216*** (7.66)	0.064*** (10.65)	0.062*** (3.21)
m1_postlehmanXcore borrowerXlog bor count	0.806*** (4.53)	0.933*** (3.39)	-0.324*** (-3.05)	-0.331*** (-3.49)
m2_postlehmanXcore borrowerXlog bor count	0.414 (1.43)	0.827** (2.40)	-0.109 (-0.63)	-0.185 (-1.20)
m3_postlehmanXcore borrowerXlog bor count	1.009*** (4.81)	1.187*** (5.23)	-0.257** (-2.05)	-0.317*** (-3.42)
q2_postlehmanXcore borrowerXlog bor count	1.464*** (11.08)	1.548*** (4.52)	-0.261*** (-3.31)	-0.340*** (-3.27)
q3_postlehmanXcore borrowerXlog bor count	2.323*** (15.40)	2.230*** (4.19)	-0.235*** (-2.60)	-0.328*** (-4.55)
q4_postlehmanXcore borrowerXlog bor count	1.027*** (6.35)	1.216*** (3.67)	-0.149 (-1.54)	-0.205*** (-2.87)
Lender FE	No	Yes	No	Yes
N	21171	21171	21171	21171

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 15: Impact of the Crisis, Periphery → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-4.398*** (-12.81)	-2.840*** (-4.55)	1.699*** (8.12)	1.650*** (7.78)
m2_postlehman	-4.525*** (-10.50)	-2.811*** (-4.31)	0.122 (0.47)	-0.098 (-0.69)
m3_postlehman	-4.995*** (-12.53)	-3.650*** (-6.10)	0.203 (0.84)	0.099 (0.58)
q2_postlehman	-4.444*** (-14.49)	-3.573*** (-6.37)	-2.027*** (-10.85)	-2.050*** (-7.07)
q3_postlehman	-3.289*** (-11.15)	-3.028*** (-4.18)	-2.152*** (-11.97)	-2.180*** (-18.08)
q4_postlehman	-3.168*** (-10.53)	-3.066*** (-7.19)	-2.839*** (-15.49)	-2.909*** (-24.81)
core borrowerXlog bor count	0.051*** (10.92)	0.138*** (5.38)	0.070*** (24.40)	0.068*** (8.54)
m1_postlehmanXcore borrowerXlog bor count	0.953*** (12.33)	0.640*** (4.46)	-0.336*** (-7.13)	-0.326*** (-6.54)
m2_postlehmanXcore borrowerXlog bor count	0.955*** (9.77)	0.634*** (4.20)	-0.135** (-2.27)	-0.086*** (-2.71)
m3_postlehmanXcore borrowerXlog bor count	1.070*** (11.78)	0.819*** (5.96)	-0.280*** (-5.05)	-0.256*** (-6.70)
q2_postlehmanXcore borrowerXlog bor count	0.979*** (14.01)	0.820*** (6.18)	-0.060 (-1.42)	-0.054 (-0.82)
q3_postlehmanXcore borrowerXlog bor count	0.726*** (10.88)	0.700*** (4.15)	-0.201*** (-4.93)	-0.192*** (-7.69)
q4_postlehmanXcore borrowerXlog bor count	0.687*** (10.02)	0.700*** (7.00)	-0.114*** (-2.74)	-0.096*** (-3.65)
Lender FE	No	Yes	No	Yes
N	86872	86872	86872	86872

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 16: Robustness I: Impact of the Crisis, Core → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-2.966*** (-3.58)	-2.999*** (-2.84)	1.141** (2.12)	1.342*** (2.83)
m2_postlehman	0.393 (0.26)	-2.042 (-1.56)	-0.559 (-0.57)	-0.222 (-0.39)
m3_postlehman	-4.329*** (-4.29)	-4.857*** (-4.99)	-0.000 (-0.00)	0.221 (0.48)
q2_postlehman	-6.605*** (-9.41)	-6.612*** (-5.03)	-1.455*** (-3.18)	-1.300*** (-3.20)
q3_postlehman	-7.293*** (-7.45)	-7.058*** (-5.78)	-2.272*** (-3.57)	-2.236*** (-7.65)
q4_postlehman	-3.141*** (-3.51)	-3.518*** (-3.83)	-2.612*** (-4.48)	-2.400*** (-6.97)
core borrowerXlog bor count	0.244*** (24.05)	0.265*** (7.98)	0.059*** (8.88)	0.056** (2.78)
m1_postlehmanXcore borrowerXlog bor count	0.671*** (3.56)	0.703*** (2.86)	-0.181 (-1.47)	-0.227** (-2.21)
m2_postlehmanXcore borrowerXlog bor count	-0.119 (-0.34)	0.506 (1.69)	0.046 (0.20)	-0.016 (-0.13)
m3_postlehmanXcore borrowerXlog bor count	1.036*** (4.46)	1.128*** (5.33)	-0.204 (-1.35)	-0.250*** (-2.93)
q2_postlehmanXcore borrowerXlog bor count	1.521*** (9.40)	1.529*** (5.10)	-0.168 (-1.60)	-0.208** (-2.27)
q3_postlehmanXcore borrowerXlog bor count	1.727*** (7.63)	1.634*** (5.84)	-0.145 (-0.99)	-0.161** (-2.63)
q4_postlehmanXcore borrowerXlog bor count	0.765*** (3.69)	0.817*** (4.04)	-0.146 (-1.08)	-0.201*** (-2.83)
Lender FE	No	Yes	No	Yes
N	16127	16127	16127	16127

t statistics in parentheses* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 17: Robustness I: Impact of the Crisis, Periphery → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-2.538*** (-7.32)	-1.420** (-2.60)	0.825*** (3.30)	0.737*** (3.02)
m2_postlehman	-2.675*** (-5.54)	-1.325** (-2.23)	-0.525 (-1.51)	-0.699*** (-4.07)
m3_postlehman	-4.014*** (-9.80)	-3.171*** (-5.53)	0.203 (0.69)	0.109 (0.52)
q2_postlehman	-3.479*** (-9.94)	-3.425*** (-5.99)	-2.676*** (-10.60)	-2.709*** (-8.03)
q3_postlehman	-1.273*** (-3.01)	-1.428* (-1.76)	-2.636*** (-8.65)	-2.651*** (-14.41)
q4_postlehman	-2.249*** (-6.59)	-2.118*** (-3.45)	-2.970*** (-12.06)	-3.024*** (-30.61)
core borrowerXlog bor count	0.173*** (37.94)	0.202*** (9.21)	0.072*** (21.80)	0.070*** (7.84)
m1_postlehmanXcore borrowerXlog bor count	0.545*** (6.92)	0.321** (2.58)	-0.136** (-2.39)	-0.117** (-2.06)
m2_postlehmanXcore borrowerXlog bor count	0.556*** (5.03)	0.299** (2.22)	0.008 (0.10)	0.046 (1.20)
m3_postlehmanXcore borrowerXlog bor count	0.872*** (9.28)	0.719*** (5.49)	-0.283*** (-4.17)	-0.263*** (-5.58)
q2_postlehmanXcore borrowerXlog bor count	0.777*** (9.64)	0.791*** (5.75)	0.089 (1.53)	0.097 (1.21)
q3_postlehmanXcore borrowerXlog bor count	0.280*** (2.86)	0.331* (1.74)	-0.092 (-1.31)	-0.086** (-2.11)
q4_postlehmanXcore borrowerXlog bor count	0.497*** (6.30)	0.483*** (3.30)	-0.088 (-1.54)	-0.073*** (-3.10)
Lender FE	No	Yes	No	Yes
N	63054	63054	63054	63054

t statistics in parentheses* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 18: Robustness II: Impact of the Crisis, Core → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-2.960*** (-3.58)	-2.993*** (-2.83)	1.140** (2.12)	1.342*** (2.83)
m2_postlehman	0.398 (0.26)	-2.037 (-1.55)	-0.559 (-0.57)	-0.222 (-0.39)
m3_postlehman	-4.326*** (-4.28)	-4.854*** (-4.99)	0.000 (0.00)	0.221 (0.48)
q2_postlehman	-6.601*** (-9.41)	-6.608*** (-5.02)	-1.455*** (-3.18)	-1.301*** (-3.20)
q3_postlehman	-7.289*** (-7.45)	-7.054*** (-5.77)	-2.272*** (-3.57)	-2.236*** (-7.64)
q4_postlehman	-3.138*** (-3.51)	-3.516*** (-3.82)	-2.612*** (-4.48)	-2.400*** (-6.96)
core borrowerXlog bor count	0.245*** (24.15)	0.266*** (8.01)	0.059*** (8.86)	0.056** (2.77)
m1_postlehmanXcore borrowerXlog bor count	0.670*** (3.56)	0.702*** (2.85)	-0.181 (-1.47)	-0.227** (-2.21)
m2_postlehmanXcore borrowerXlog bor count	-0.120 (-0.35)	0.505 (1.68)	0.046 (0.20)	-0.016 (-0.13)
m3_postlehmanXcore borrowerXlog bor count	1.035*** (4.46)	1.128*** (5.33)	-0.204 (-1.35)	-0.250*** (-2.93)
q2_postlehmanXcore borrowerXlog bor count	1.520*** (9.40)	1.529*** (5.10)	-0.168 (-1.60)	-0.208** (-2.26)
q3_postlehmanXcore borrowerXlog bor count	1.726*** (7.62)	1.633*** (5.83)	-0.145 (-0.99)	-0.161** (-2.62)
q4_postlehmanXcore borrowerXlog bor count	0.764*** (3.69)	0.816*** (4.04)	-0.146 (-1.08)	-0.201*** (-2.83)
Lender FE	No	Yes	No	Yes
N	16119	16119	16119	16119

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 19: Robustness II: Impact of the Crisis, Periphery → Core Links

	Amount		Rate	
	(1)	(2)	(3)	(4)
m1_postlehman	-2.334*** (-6.87)	-1.329** (-2.40)	0.859*** (3.40)	0.756*** (3.05)
m2_postlehman	-2.671*** (-5.67)	-1.282** (-2.15)	-0.552 (-1.58)	-0.698*** (-4.05)
m3_postlehman	-3.835*** (-9.54)	-3.155*** (-5.42)	0.191 (0.64)	0.081 (0.39)
q2_postlehman	-3.388*** (-9.92)	-3.380*** (-5.87)	-2.675*** (-10.55)	-2.709*** (-8.00)
q3_postlehman	-1.243*** (-3.02)	-1.401* (-1.71)	-2.639*** (-8.62)	-2.650*** (-14.38)
q4_postlehman	-2.217*** (-6.66)	-2.098*** (-3.40)	-2.972*** (-12.02)	-3.024*** (-30.53)
core borrowerXlog bor count	0.172*** (38.12)	0.210*** (9.47)	0.070*** (20.87)	0.070*** (7.69)
m1_postlehmanXcore borrowerXlog bor count	0.503*** (6.51)	0.300** (2.38)	-0.143** (-2.50)	-0.122** (-2.10)
m2_postlehmanXcore borrowerXlog bor count	0.562*** (5.22)	0.289** (2.14)	0.015 (0.19)	0.046 (1.19)
m3_postlehmanXcore borrowerXlog bor count	0.837*** (9.06)	0.716*** (5.37)	-0.280*** (-4.07)	-0.256*** (-5.44)
q2_postlehmanXcore borrowerXlog bor count	0.763*** (9.70)	0.781*** (5.63)	0.089 (1.53)	0.097 (1.20)
q3_postlehmanXcore borrowerXlog bor count	0.280*** (2.94)	0.325* (1.70)	-0.091 (-1.29)	-0.086** (-2.11)
q4_postlehmanXcore borrowerXlog bor count	0.497*** (6.46)	0.480*** (3.26)	-0.087 (-1.52)	-0.073*** (-3.09)
Lender FE	No	Yes	No	Yes
N	61239	61239	61239	61239

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

1.6 Appendix: Core-Periphery Fitting Model

To fit the Core-Periphery network structure on the Italian Overnight market we adopt the iterative algorithm proposed by Craig and Von Peter (2014). The method the authors propose is based on comparing a network to a block model that consists of a fully-connected core and a periphery that has no internal edges but is fully connected to the core. The matrix so defined, can be seen as a generalization of the star network, i.e. extreme case of a centralized network. In such ideal pattern, there exists a single node in the center of the network, which is connected to all other nodes, where the latter are not connected to each other. Craig and Von Peter (2014) generalize such pattern allowing for multiple nodes to be in the center of the network.

In block modeling terms this is means to specify an adjacency matrix, such that connections between cores are represented by a region containing only ones (CC), whereas the of connections between periphery banks is represented by a region of the matrix with only zeros PP . Off-diagonal regions in the matrix CP and PC are represented by imperfect 1-blocks:

$$M = \begin{pmatrix} 1 & CP \\ PC & 0 \end{pmatrix} \quad (3)$$

The authors imposing furthermore that CP and PC regions in the ideal pattern of Core-Periphery structure to be respectively row-regular and column regular, use a combinatorial optimization technique to maximize the pattern matrix induced by the partition and reduce the errors between the fit of the analyzed network and matrix M . The fitting procedure is divided into two steps: first, they define a measure of distance $e(C)$ between the observed network and the ideal pattern of Core-Periphery and then, solve for the optimal partition of banks into core and periphery that minimizes the residuals deriving by the errors between the observed network and the idealized pattern.

Given $e(C)$, it is possible to count the number of errors generated by the algorithm for each chosen network partition. The optimal core will be represented by the partition C yielding the smallest number of inconsistencies. The number of errors can then be normalized

by the total number of links ($N - N_c$) in the given network:

$$e(C) = \frac{E_{cc} + E_{pp} + (E_{cp} + E_{pc})}{\sum_i \sum_j a_{ij}} = \frac{E_{cc} + E_{pp}}{A} \quad (4)$$

where E_{cc} , E_{pp} , E_{cp} and E_{pc} represent the number of inconsistencies in the four matrix regions generated by partition C with respect to the ideal Core-Periphery pattern. Given the previous definition, the optimal partition C^* will be equal to:

$$C^* = \operatorname{argmine}(C) = C \in \Gamma | e(C) \forall c \in \Gamma \quad (5)$$

where Γ , indicates the number and identity of banks identified as cores.

2 Chapter: Portfolio Diversification and Systemic Risk in Interbank Networks

with Stefano Battiston (University of Zurich) and Paolo Tasca (University College London)

Abstract

This paper contributes to a growing literature on the ambiguous effects of risk diversification. In our model, banks hold claims on each other's liabilities that are marked-to-market on the individual financial leverage of the obligor. The probability of systemic default is determined using a passage-problem approach in a network context and banks are able to internalize the network externalities of contagion through their holdings. Banks do not internalize the social costs to the real economy of a systemic default of the banking system. We investigate the optimal diversification strategy of banks in the face of opposite and persistent economic trends that are ex-ante unknown to banks. We find that the optimal level of risk diversification may be interior or extremal depending on banks exposure to the external assets and that a tension arises whereby individual incentives favor a banking system that is over-diversified with respect to the level of diversification that is desirable in the social optimum.

2.1 Introduction

The folk wisdom of “not putting all of your eggs in one basket” has been a dominant paradigm in the financial community in recent decades. Pioneered by the works of Markowitz (1952), Tobin (1958) and Samuelson (1967), analytic tools have been developed to quantify the benefits derived from increased risk diversification. However, recent theoretical studies

have begun to challenge this view by investigating the conditions under which diversification may have undesired effects (see, e.g., Battiston et al., 2012b; Ibragimov et al., 2011; Wagner, 2011; Stiglitz, 2010; Brock et al., 2009; Wagner, 2010; Goldstein and Pauzner, 2004). These works have found various types of mechanisms leading to the result that full diversification may not be optimal. For instance, Battiston et al. (2012b) assume an amplification mechanism in the dynamics of the financial robustness of banks; Stiglitz (2010) assumes that the default of one actor implies the default of all counterparties.

Our paper is closely related to Wagner (2010). They study how the trade-off between diversification and the diversity of portfolio held by different banks may lead to full diversification being suboptimal from society's perspective. The underlying assumption of their model is that in the case of a single default, the insolvent bank can sell its assets to the solvent bank and avoid physical (and more costly) liquidation. Since liquidation costs are not internalized by the banks, payoff maximizing and social optimal levels of diversification not necessarily coincide. This implies the existence of negative externalities among the banks, whereby by increasing a bank's diversification level it also increases the possibility of costly liquidation of assets by the other, therefore augmenting the joint failures of systemic crises.

Differently from Wagner and from the aforementioned literature, this paper sheds light on a new mechanism under which a tension may arise between individual risk diversification and systemic risk. We demystify the effects of diversification in a complex context, where banks not only hold overlapping portfolios but also hold claims on each other liabilities. In this setting, a bank's payoff not only depends on the bank own financial condition, but also on the financial conditions of the other banks to which it is interconnected. We model the default of a bank as a problem of first passage time in a network-based stochastic diffusion process where the valuation of a bank liabilities relies also on the liabilities of the other banks in its network. We can show that the ambiguous role of diversification on systemic risk is much more complex and pronounced than in the setting proposed by Wagner and it involves an amplification of losses along the chain of lending relations, which - depending on the interbank network configuration - may lead the optimal degree of diversification not to

be internal, as in Wagner, but to be at the corner: either minimum or maximum.

In particular, this paper assumes an arbitrage-free market where price returns are normally distributed and uncorrelated. Moreover, there are negative externalities arising from the fact that banks' interbank assets are negatively affected when a counterparty's leverage increases. Banks internalize these externalities since their utility function is computed using a default probability model that accounts for the contagion from counterparties. However, the market may follow positive or negative trends that are ex-ante unpredictable and persist over a certain period of time. This incomplete information framework leads to a problem of portfolio diversification under uncertainty. In fact, portfolio returns display a bimodal distribution resulting from the combination of two opposite trends weighted by the probability of being either in a bad or in a good state of the world. We find that optimal diversification can be *interior*. This result holds both at the *individual* and at the *social* welfare level. Moreover, we find that individual incentives favor a financial system that is over-diversified with respect to what is socially efficient.

More in detail, we consider a banking system composed of *leveraged* and risk-averse financial institutions (hereafter, "banks") that invest in two asset classes. The first class consists of debts issued by other banks in the network (hereafter, "interbank claims"). The value of these securities depends, in turn, on the leverage of the issuers. The second class represents risky assets that are external to the financial network and may include, e.g., mortgages on real estate, loans to firms and households and other real economy-related activities (hereafter, "external assets"). The underlying economic cycle is the primary source of external asset price fluctuations, but it is unknown *ex ante* to the banks and, with a certain probability p , it may be positive (hereafter, "uptrend") or negative (hereafter, "downtrend"). In this paper, we focus on the effect of varying levels of diversification across external assets, while the interbank network is considered as given.

As a general property, diversification of idiosyncratic risks lowers the volatility of a bank's portfolio of external assets and increases the likelihood of the portfolio to follow the economic trend underlying the price movements. Therefore, if the future economic trend is unknown

and banks cannot divest for a certain period, risk diversification is beneficial when the economic trend happens to be positive, because it reduces the *downside risk*. In contrast, risk diversification is detrimental when the economic trend happens to be negative because it reduces the *upside potential*. As a result, the intuition would suggest that the optimal level of risk diversification is always interior and depends on the probability of the market trend to be positive or negative. However, this intuition arises from the logical fallacy that maximizing a convex combination of functions is equivalent to take the convex combination of the maxima, which is not correct in general.

As a first result, we show that optimal risk diversification is interior under certain conditions, but is not interior in general. In particular, it can be optimal for individual banks to pursue a full diversification strategy even when the downtrend is almost as probable as the uptrend. Because full diversification implies that all banks are exposed to the same shocks, the probability of a systemic default conditional to an individual default tends to one. This fact leads to our second result: in a wide range of parameters there exists a tension between the individual bank's incentive to fully diversify and social optimum due to social costs associated to simultaneous defaults. Interestingly, the tension exists both when the optimal is interior and when it is an extreme point of the feasible range of m .

The network of interbank claims exposes banks to shocks on external assets held by their counterparties. Therefore, the network amplifies the effect of the negative trend, when it occurs, and the impact of shocks when banks have largely overlapping portfolios due to extensive diversification on external assets.¹⁴

2.1.1 Related work

One of the novelties of our work is the fact that the result about interior optimal diversification holds even in the absence of asymmetric information, behavioral biases or transaction costs and taxes. Moreover, we do not need to impose *ad hoc* asset price distributions as in the literature on diversification pitfalls in portfolios with fat-tailed distributions (to name a

¹⁴Tasca (2013) proposed a mathematical relation between portfolio diversification and the level of overlapping (and correlation) between two portfolios.

few, Zhou, 2010; Ibragimov et al., 2011; Mainik and Embrechts, 2012).

In our model, because external assets carry idiosyncratic risks, banks have an incentive to diversify across them. In this respect, similar to Evans and Archer (1968); Statman (1987); Elton and Gruber (1977); Johnson and Shannon (1974), we measure how the benefit of diversification vary as the external assets in an equally weighted portfolio is increased. This benchmark is the so-called $1/n$ or *naive* rule. However, because banks are debt financed, we depart from the methods of those previous studies by modeling risk not in terms of a portfolio's standard deviation but in terms of the *default probability*. Indeed, in that literature the relationship between default probability and portfolio size has not been investigated in depth.

In order to investigate the notion of default probability in a system context we develop a framework in which banks are connected in a network of liabilities, similarly to the stream of works pioneered by Eisenberg and Noe (2001). However, that literature considers only the liquidation value of corporate debts at the time of default. In particular, in the works based on the notion of “clearing payment vector” (e.g., Cifuentes et al., 2005; Elsinger et al., 2006), the value of interbank claims depends on the solvency of the counterparties at the maturity of the contracts and it is determined as the fixed point of a so-called “fictitious sequential default” algorithm. Starting from a given *exogenous* shock on one or more banks, one can measure *ex-post* the impact of the shock in the system and investigate, for instance, which structure are more resilient to systemic risk (Battiston et al., 2012a; Roukny et al., 2013).

Our objective instead is to derive the default probability of individual banks, in a system context, that can be computed by market players *ex-ante*, i.e. before the shocks are realized and before the maturity of the claims. A related question was addressed in (Shin, 2008) where one assumes that asset values are random variables that move altogether according to a same scaling factor. The expected value of the assets is plugged into the Eisenberg-Noe fixed point algorithm yielding an estimate of the values of the liabilities before the observation of the shocks. However, the latter approach does not apply if assets are independent random variables and, more importantly, it does not address the issue of how the default probability

of the various banks are related.

Strictly speaking, default means that the bank is not able to meet its obligations at the time of their maturity. Therefore, in principle, it does not matter whether, any time before the maturity of the liabilities, the total asset value of a firm falls beneath the book value of its debts as long as it can recover by the time of the maturity. In practice, however, it does matter a lot. This is the case, for instance, if the bank has also some short-term liabilities and short-term creditors decide to run on the bank. Indeed there is a whole literature that building on Black and Cox (1976) investigates the notion of *time to default* in various settings. Such notion extends the framework of Merton (1974) by allowing defaults to occur at any *random* time before the maturity of the debt, as soon as the firm assets value falls to some prescribed lower threshold.

Drawing inspiration from these approaches, in this paper we model the evolution over time of banks assets as stochastic processes where, at the same time, the value of interbank claims is a function of the financial leverage of the counterparties as reflected by the credit-liability network. Although from a mathematical point of view, the framework requires to deploy the machinery of continuous stochastic processes, this work offers a valuable way to compute the default probability in system context under mild assumptions. The default probability can be written in analytical form in simple cases and it can be computed numerically in more complicated cases. An underlying assumption in the model is to consider the credit spread of counterparties as an increasing function of their leverage, i.e. the higher the leverage the higher the credit spread. As a benchmark, in this paper we assume that such a dependence is linear.

In general, the framework developed here allows to investigate how the probability of defaults depends on certain characteristics of the network such as the number of interbank contracts and the number of external assets. In this paper, we focus on the diversification level across external assets and we look at the limit in which analytical results can be obtained. The assumption we make is that the interbank market is relatively tightly knit and banks are sufficiently homogeneous in balance sheet composition and investment strategies.

Indeed, it has been argued that the financial sector has undergone increasing levels of homogeneity, Haldane (2009). Moreover, empirical evidence shows that bank networks feature a core-periphery structure with a core of big and densely connected banks and a periphery of smaller banks. Thus, our hypothesis of homogeneity applies to the banks in such a core (see, e.g., Elsinger et al., 2006; Iori et al., 2006; Battiston et al., 2012c; Fricke and Lux, 2012).

The paper is organized as follows. In Section 2.2, we introduce the model. Section 2.3 adopts a marginal benefit analysis by formalizing the single bank utility maximization problem with respect to the number of external assets in the portfolio. In Section 2.4, we compare private incentives of risk diversification with social welfare effects. Section 2.5 concludes the paper and considers some policy implications.

2.2 Model

Let time be indexed by $t \in [0, \infty]$ in a system of N risk-averse leveraged banks with mean-variance utility function. To ensure simplicity in notation, we omit the time subscripts whenever there is no confusion. For the bank $i \in \{1, \dots, N\}$, the balance sheet identity conceives the equilibrium between the asset and liability sides as follows:

$$a_i = l_i + e_i, \quad \forall t \geq 0 \quad (1)$$

where $\mathbf{a} := (a_1, \dots, a_N)^T$ is the column vector of bank assets at market value. $\mathbf{l} := (l_1, \dots, l_N)^T$ is the column vector of bank debts at book face value. There is an homogeneous class of debt with maturity T and zero coupons, i.e., defaultable zero-coupon bonds. $\mathbf{e} := (e_1, \dots, e_N)^T$ is the column vector of equity values. Notably, the market for investment opportunities is complete and composed of two asset classes that are perfectly divisible and traded continuously: (i) N interbank claims, and (ii) M external assets related to the real side of the economy. There are no transaction costs or taxes. However, there are borrowing and short-selling restrictions. Each bank selects a portfolio composed of $n \leq N - 1$ interbank claims

and $m \leq M$ external assets. Then, the asset side in Eq. (1) can be decomposed as¹⁵:

$$a_i := \sum_j z_{ij} \nu_j + \sum_k w_{ik} \hat{\eta}_k. \quad (2)$$

In the above equation $\mathbf{Z} := [z_{ij}]_{N \times M}$ is the $N \times M$ matrix of external investments in which each entry $z_{ij} \geq 0$ is the number of units of external asset j at price ν_j held by bank i . $\mathbf{W} := [w_{ik}]_{N \times N}$ is the $N \times N$ (right) stochastic matrix, i.e. the coefficient in the (i,k) cell of \mathbf{W} , $w_{ik} \in [0, 1]$ indicates the fraction of borrowing by i from k . We suppose that interbank claims, denoted by $\hat{\eta}_k$, are marked-to-market and are priced according to the discounted value of future payoffs at maturity:

$$\hat{\eta}_i = \frac{\eta_i}{(1 + r_i)^{T-t}} \quad (3)$$

where η_i denotes the book value of bank i 's obligations towards other banks in the network, i.e. the value of the payment at maturity that i promised to its bond holders. r_i is the rate of return on $(T - t)$ -years maturity obligations. $c_i = r_i - r_f$ is the credit spread (premium), over the risk free rate r_f , paid by i to the bond holders.

In a stylized form, each bank's balance-sheet is as follows:

Bank- i balance-sheet

Assets	Liabilities
$\sum_k w_{ik} \hat{\eta}_k$	l_i
$\sum_j z_{ij} \nu_j$	e_i

¹⁵Notice that, for the sake of simplicity, we omit the lower and upper bounds of the summations. It remains understood that, in the summation for external assets, the index ranges from 1 to M , and that, in the summation for banks, the index ranges from 1 to N (with the condition that $w_{ii} = 0$ for all $i \in \{1, \dots, N\}$).

2.2.1 Leverage and Default Event

Our approach to define the default in the same spirit of Black and Cox (1976), who extends Merton (1974) by allowing for a premature default when the asset value of the firm falls beneath the book value of its debt. From a technical point of view, what matters is the debt-to-asset ratio:

$$\phi_i := \frac{l_i}{a_i}, \quad (4)$$

with natural bound $[\varepsilon, 1]$ where $\begin{cases} 1 & \text{default boundary} \\ \varepsilon \rightarrow 0^+ & \text{safe boundary} \end{cases}$.

Definition of Default Event: The probability of the default event is the probability that ϕ_i , initially at an arbitrary level $\phi_i(0) \in (\varepsilon, 1)$, exits for the first time through the *default boundary* 1, after time $t > 0$. More precisely, we use the concept of first exit time, τ , through a particular end of the interval $(\varepsilon, 1)$.¹⁶ Namely,

$$\tau := \inf \{t \geq 0 \mid \mathbf{1}_{\phi_i(t) \leq \varepsilon} + \mathbf{1}_{\phi_i(t) \geq 1} \geq 1\}. \quad (5)$$

If the default event is defined as $(default_i) := \{\phi_i(\tau) \geq 1\}$, then the default probability is the probability of this event:

$$\mathbb{P}(default_i) = \mathbb{P}(\phi_i(\tau) \geq 1).$$

With a slight abuse of notation we rewrite $\mathbb{P}(default_i)$ as:

$$\mathbb{P}(default_i) = \mathbb{P}(\phi_i \geq 1). \quad (6)$$

¹⁶The notion of exit time is a crucial quantity in stochastic processes. Such notion has the important advantage to link the default event of the bank with structural (observable) variables of the bank. Hence, default event becomes a predictable event corresponding to the cases where the value of bank asset is too close to the value of its debt.

Leverage in System Context: Combining together Eq. (2)-(3)-(4), we obtain:

$$\phi_i = l_i / \left(\sum_j z_{ij} \nu_j + \sum_k w_{ik} \left(\frac{\eta_k}{(1+r_k)^{(T-t)}} \right) \right). \quad (7)$$

In general terms, the credit spread of bank i depends on several factors such as the firm's leverage, the volatility of the underlying assets or the liquidity risk (see, e.g., Collin-Dufresne et al., 2001). In studying Eq. (6) in a system context, in order to better isolate the explanatory power of leverage, here we assume that the credit spread depends only on leverage in a linear fashion:

$$c_i = \beta \phi_i. \quad (8)$$

In reality, the relation between credit spread and leverage can be more complicated. The rationale adopted in our paper is that the main determinant of the credit spread must be the distance to default, measured by the ratio c_i of debt over total asset, thus the credit spread must be a function f of such ratio: $c_i := \beta f(\phi_i)$. It must be that the closer ϕ_i to one, the higher the probability of default and thus the higher the credit spread c_i . In other words, f should be non decreasing. Moreover, because of the endogenous dynamics of asset prices, the function f is likely to be highly non-linear. Since it is not possible anyway to derive it in closed form, in our paper, for the sake of simplicity, we take a parsimonious first order linear approximation, $c_i := \beta f(\phi_i)$, which serves as a conceptually simple approximation of the original unknown function f . This assumption allows us to capture the main basic feature that we expect from a credit spread, namely to increase with the default probability. The parameter $\beta (> 0)$ is the factor loading on i 's leverage ϕ_i and can be understood as the responsiveness of the rate of return to the leverage. Then, by replacing Eq. (8) into Eq. (3) we have:

$$\hat{\eta}_i = \frac{\eta_i}{(1+r_f + \beta \phi_i)} \quad (9)$$

where, w.l.g. $T - t = 1$. This means that banks issue 1-year maturity obligations that are continuously rolled over. Notice that, by Eq. (6) and Eq. (9), even in the case of a high default probability, bank debts are still priced at a positive market value. Namely, for $\phi_i \rightarrow 1$, $\hat{\eta}_i > 0$. This means that, creditors are assumed to partially recover their credits in case of default.

REMARK 1. *If for $\phi_i \rightarrow 1$ the marked-to-market value of interbank liabilities is such that $\hat{\eta}_i > 0$, the implicit recovery rate is:*

$$\delta = \begin{cases} 1 & \text{in case of no default} \\ \frac{1+r_f}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

Proof: 2.6.

Now, by using Eq. (9) we can rewrite Eq. (7) as:

$$\phi_i = l_i / \left(\sum_j z_{ij} \nu_j + \sum_k w_{ik} \left(\frac{\eta_k}{1 + r_f + \beta \phi_k} \right) \right). \quad (10)$$

Eq. (10) highlights a *non-linear* dependence of ϕ_i from the leverage $\phi_{k=1,\dots,n}$ of the other banks to whom i is exposed via the matrix \mathbf{W} ¹⁷.

Recent works based on the “clearing payment vector” mechanism (see, e.g., Eisenberg and Noe, 2001; Cifuentes et al., 2005) provide a “fictitious sequential default” algorithm to determine the liquidation equilibrium value of interbank claims at their maturity. In reality, however, defaults may happen before the maturity of the debts. In this respect, Eq. (9) together with Eq. (10) captures, even before the maturity of the debts, the market value of interbank claims in the building up of the distress spreading from one bank to another.

REMARK 2. *Bank’s leverage defined as in Eq. 10 is a second order polynomial equation in Φ :*

$$\Phi \mathbf{I} \beta \Phi + \Phi \mathbf{I} \mathbf{R} - \mathbf{L}(\mathbf{W} \mathbf{H})^{-1} \beta \Phi + \Phi = \mathbf{L}(\mathbf{W} \mathbf{H})^{-1} \mathbf{R} \quad (11)$$

¹⁷The obligation of each bank i can be considered as an n -order derivative, the price of which is derived from the risk-free rate r_f and from the leverage ϕ_i of i . The latter, in turn, depends on the leverage $\phi_{k=1,\dots,n}$ of the other banks obligors of i .

where $\mathbf{H} := \text{diag}(\eta_1, \eta_2, \dots, \eta_M)$; $\mathbf{I} := \mathbf{ZV}(\mathbf{WH})^{-1}$ and $\mathbf{\Phi} := \text{diag}(\phi_1, \phi_2, \dots, \phi_N)$; $\mathbf{L} := \text{diag}(l_1, l_2, \dots, l_N)$; $\mathbf{V} := \text{diag}(\nu_1, \nu_2, \dots, \nu_M)$; $\mathbf{R} := \text{diag}(R, R, \dots, R)$ with $R = 1 + r_f$; $\mathbf{W} := [w_{ik}]_{N \times N}$; $\mathbf{Z} := [z_{ij}]_{N \times M}$.

Proof: 2.6.

Along this line of reasoning, one can notice that the default probability of a given bank depends on the likelihood of its leverage to hit the default boundary. This, in turn, depends on the joint probability of the other banks' leverages, to whom this banks is connected, of hitting the default boundary. In order to account for these network effects, in the next section we will provide an explicit form of default probability in system context.

2.3 Benefits of Diversification in External Assets

Similar to Evans and Archer (1968); Statman (1987); Elton and Gruber (1977); Johnson and Shannon (1974); Bird and Tippett (1986), in this section we measure the advantage of diversification by determining the rate at which risk reduction benefits are realized as the number $m (\leq M)$ of external assets in an *equally weighted* portfolio is increased. In contrast with those studies, rather than minimizing the variance of the banks' assets, we maximize their expected utility with respect to m . The methodology is explained in the following subsections. In 2.3.1 we formalize the equally weighted portfolio of external assets. 2.3.2 defines the systemic default event. In 2.3.4 we formalize the bank utility function and in 2.3.5 we maximize the utility function with respect to the control variable m .

2.3.1 Equally Weighted Portfolio of External Assets

From Eq. (2), let bank i 's portfolio of external assets be defined as:

$$s_i := \sum_j z_{ij} \nu_j. \quad (12)$$

To study the benefits of diversification, in isolation, we need to consider the $1/m$ (equally weighted) portfolio allocation. As the risk of a stock portfolio depends on the proportions of

the individual stocks, their variances, and covariances, our assumption implies that the level of portfolio risk will decrease with the number of external assets m . Thus, every equally weighted and uncorrelated asset added to a portfolio will increase the level of diversification and reduce the portfolio risk. This allocation is adopted as a metric to measure the rate at which risk-reduction benefits are realized as the number of assets held in the portfolio is increased.

Recall that we assume that the projects have the same average return and variance. In order for the balance-sheet identity to hold true at any time t for the asset side, the fraction of assets allocated to each project must be kept constant over time (see eg. Maillard et al. (2010); Tasca et al. (2014)). In other words, because the market price of external assets $s(t)$ can change for market reasons, the bank has to adjust the number of units of external assets in its portfolio with respect to the price changes in order to keep the level of investment in each external asset at $s_i(t)/n_i$. Consistently with the assumption of such portfolio rebalancing strategy of banks, we assume that external assets are equally weighted in banks' portfolios.

Formally, for every external asset $j \in \{1, \dots, M\}$ and each bank $i \in \{1, \dots, N\}$, the fraction of portfolio s_i invested by bank i in the external asset j is:

$$\frac{1}{m} = \frac{z_{ij} \nu_j}{s_i} . \quad (13)$$

The minimum conditions that allow us to apply the $1/m$ rule, as a benchmark, without violating the mean-variance dominance criterion, is to assume the external assets are statistically indistinguishable, i.e., they have the same drift, the same variance and they are uncorrelated.¹⁸

Thus, the price of external assets is properly characterized by following time-homogeneous

¹⁸Notice that under those conditions, the $1/m$ portfolio allocation is Pareto optimal. See e.g., Rothschild and Stiglitz (1971); Samuelson (1967); Windcliff and Boyle (2004).

diffusion process:¹⁹

$$\frac{d\nu_j}{\nu_j(t)} = \mu dt + \sigma d\tilde{B}_j(t), \quad j = 1, \dots, M \quad (14)$$

Using the expression in Eq. (13), we arrive after some transformations at the following dynamics for the the portfolio in Eq. (12):

$$\frac{ds_i}{s_i(t)} = \mu dt + \frac{\sigma}{\sqrt{m}} dB_i(t). \quad (15)$$

$B_i = \frac{1}{m} \sum_j \tilde{B}_j$ is an equally weighted linear combination of Brownian shocks s.t. $dB_j \sim N(0, dt)$.

Properties of the Portfolio: There exist two states of the world, $\theta = \{0, 1\}$. This captures a situation in which the economy is either in a boom ($\theta = 1$) or in a bust ($\theta = 0$) state and is reminiscent of a stylized economic cycle. The probability that the world is in state θ is denoted as $\mathbb{P}(\{\theta\})$ with $\mathbb{P}(\{0\}) = p$ and $\mathbb{P}(\{1\}) = 1 - p$. According to the state of the world, the market of external assets is assumed to follow a given constant stochastic trend under a certain probability space $(\Omega_\mu, \mathcal{A}, \mathbb{P})$. The sample space $\Omega_\mu = \{\mu^-, \mu^+\}$ is the set of the outcomes. We use the convention:

$$\begin{cases} \mu < 0 := \mu^- & \text{if } \theta = 0, \\ \mu \geq 0 := \mu^+ & \text{if } \theta = 1 \end{cases}$$

with $|\mu^+| = |\mu^-|$. The σ -algebra \mathcal{A} is the power set of all the subsets of the sample space, $\mathcal{A} = 2^{\Omega_\mu} = 2^2 = \{\{\mu^-\}, \{\mu^+\}, \{\mu^-, \mu^+\}, \{\}\}$ is the probability measure, $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$ with $\mathbb{P}(\{\}) = 0$, $\mathbb{P}(\{\mu^-\}) = p$ and $\mathbb{P}(\{\mu^+\}) = 1 - p$. That is, p and $(1 - p)$ are the probabilities of having a downtrend and an uptrend, respectively.

To conclude, portfolio returns display a mixture distribution expressed by the convex

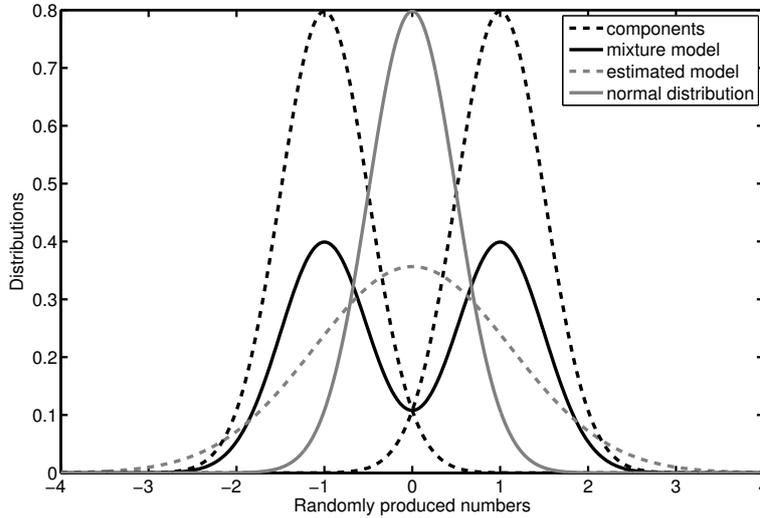
¹⁹Where $\tilde{B}_j(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$, with $\mathcal{F}_t = \sigma_y\{\tilde{B}(s) : s \leq t\}$, μ is the instantaneous risk-adjusted expected growth rate, $\sigma > 0$ is the volatility of the growth rate and $\mathbb{E}(d\tilde{B}_j, d\tilde{B}_y) := \rho_{jy} = 0$.

combination of two normal distributions weighted by p and $1 - p$. Namely,

$$\frac{ds_i}{s_i} \sim pN\left(\mu^-, \frac{\sigma}{\sqrt{m}}\right) + (1-p)N\left(\mu^+, \frac{\sigma}{\sqrt{m}}\right) \quad (16)$$

$$\text{with } \begin{cases} \mathbb{E}\left[\frac{ds_i}{s_i}\right] := \hat{\mu} = p\mu^+ + (1-p)\mu^- \\ \mathbb{E}\left[\left(\frac{ds_i}{s_i} - \hat{\mu}\right)^2\right] := \hat{\sigma}^2 = p\left[(\mu^+ - \hat{\mu})^2 + \frac{\sigma^2}{m}\right] + (1-p)\left[(\mu^- - \hat{\mu})^2 + \frac{\sigma^2}{m}\right]. \end{cases}$$

Figure 1: Distribution of portfolio returns (mixture model).



Comparison between a portfolio with returns following a bimodal distribution (black color) and a portfolio with normally distributed returns (gray colour). Parameters: $\mu = 0$, $\mu^- = -1$, $\mu^+ = 1$, $\sigma = 0.5$, $p = 0.5$.

Figure 1 illustrates this result by comparing two probability density functions (pdf). The first (gray color) curve represents the pdf of a portfolio with returns X normally distributed, i.e., $X \sim N(0, 1)$. The second (black color) curve represents the pdf of a portfolio with returns distributed as $Y1 \sim N(1, 0.5)$ with probability $p = 0.5$ and as $Y2 \sim N(-1, 0.5)$ with probability $1 - p = 0.5$. Notice that in a world where the market displays a normal distribution, the middle part of the distribution range (the “belly”) is the most likely outcome. In contrast, in a bimodal world, the belly is the least likely outcome. Moreover,

the tails of the bimodal distribution are higher than those of the normal distribution. This indicates the higher probability of severe left and right side events. The distance between the two peaks depends on the difference between the means of the two normal distributions, $|(\mu^+) - (\mu^-)|$. The (possible) asymmetry between the two peaks and the skewness of the bimodal distribution depends on the difference between the probability of having an uptrend or a downtrend, $|(1 - p) - p|$. As a result, portfolio diversification choice is subject to much more uncertainty in a bimodal world than in a normal one. The next sections show how the statistical properties of the bimodal distribution impact on the risk diversification effects.

2.3.2 Systemic Default Probability

Interbank Network structure and Contagion The model described above can be used to address several questions. In this paper, we analyze the properties of the model in a specific scenario.

It has been argued that the financial sector has undergone increasing levels of homogeneity, Haldane (2009). Moreover, empirical evidence shows that bank networks feature a core-periphery structure with a dense core of fully connected banks and a periphery of small banks. Thus, an hypothesis of homogeneity is realistic at least for the banks in the core (see, e.g., Elsinger et al., 2006; Iori et al., 2006; Battiston et al., 2012c).

We therefore assume that (i) banks have homogeneous capital structure, (ii) hold similar balance sheets and portfolios, and (iii) hold claims on each other liabilities in a *tightly knit* interbank network.

Furthermore, we do not consider here issues related to endogenous interbank network formation, optimal interbank network structures and network efficiency. See, e.g. Gale and Kariv (2007), Castiglionesi and Navarro (2007) and the survey by Allen and Babus (2009) for discussion of these topics.

Relation between Individual and Systemic Default Under the assumption of homogeneity, banks have the same capital structure:

- the portfolio of external assets is similar across banks, i.e., $s_i = s$ for all $i \in \{1, \dots, N\}$;
- the book value of promised payments at maturity is equal for every bank, i.e., $l_i = l$ for all $i \in \{1, \dots, N\}$.

The last assumption implies a symmetry between liabilities of bank i , l_i and the face value of its interbank assets, η_k in Eq. (10). Under those conditions, leverage levels among banks are equal.²⁰ Namely, the leverage of every bank will be equal to the *market leverage*:

$$\phi_i = \frac{1}{N} \sum_j^N \phi_j := \phi ,$$

for all $i \in \{1, \dots, N\}$. The above assumption, together with the assumption that $l_i \equiv \eta_k \equiv l$ and the results from the previous section, allow us to rewrite Eq. (10) as:

$$\phi = l / \left(s + w \frac{l}{1 + r_f + \beta \phi} \right). \quad (17)$$

where $w \in [0, 1]$ indicates the fraction of borrowing of banks in the interbank.

Eq. (17) is a quadratic expression in ϕ :

$$\phi^2 \beta s + \phi (sR + l(1 - \beta)) - wlR = 0 , \quad R = 1 + r_f . \quad (18)$$

Numerically, we find that Eq. (18) has always one positive and one negative root for any values of the parameters ($\beta > 0$, $R \geq 1$, $s > 0$, $l > 0$) in their range of variation:

$$\begin{cases} \phi_{pos} &= \frac{1}{2\beta s} [l(\beta - 1) - Rs + (4\beta l R s w + (l(1 - \beta) + Rs)^2)^{1/2}] , \\ \phi_{neg} &= \frac{1}{2\beta s} [l(\beta - 1) - Rs - (4\beta l R s w + (l(1 - \beta) + Rs)^2)^{1/2}] . \end{cases}$$

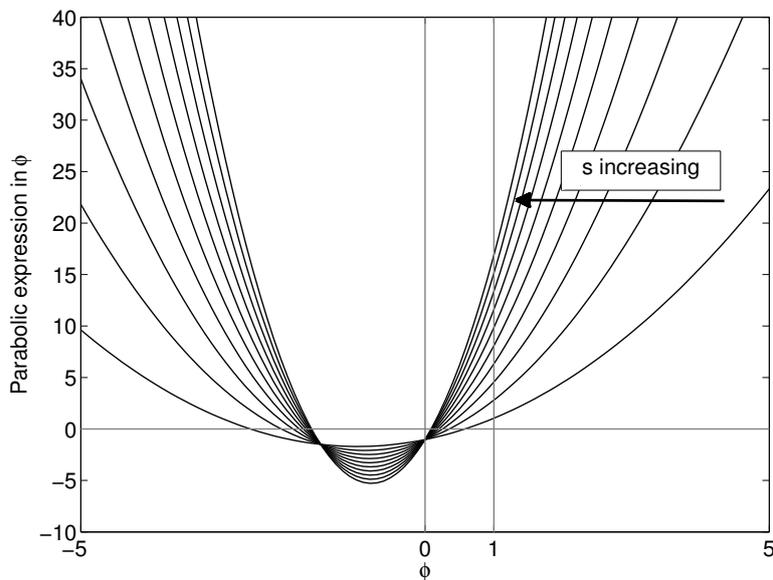
²⁰The validity of such approximation increases with the level of homogeneity across banks. Because the empirical literature consistently reports the existence of densely interconnected core of banks in interbank markets, the assumption intends to capture situations like the one of 2007-2009, in which a core of banks was highly connected and heavily exposed to the same mortgage backed securities. This operation on one hand makes the variables homogeneous, but allows us, on the other hand, to explore the systemic effects arising from banks' interconnections while being parsimonious on the assumptions.

Since, by definition ϕ can only be positive, we exclude the negative solution. Therefore, one can always find a unique positive solution to Eq. (18):

$$\phi := \phi_{pos}. \quad (19)$$

The parabola in Eq. (18) is depicted in Figure 2. For different values of s the roots are those points crossing zero in the interval $[\varepsilon, 1]$, which is the natural bound of the leverage, see Eq. (4).

Figure 2: Parabolic expression of the market leverage.



The market leverage is the solution of a parabolic expression that implies a non linear relationship between banks' leverage. Parameters: $l = 0.9$, $s \in \{1, 2, \dots, 10\}$, $r_f = 0.03$, $\beta = 0.5$, $w = 1$.

The setting described above motivates the assumption that the leverage of the individual bank approximates the leverage of the system:

$$\phi_i \simeq \phi \quad (20)$$

for all $i \in \{1, \dots, N\}$. Then, if we apply Eq. (6), for the Continuous Mapping Theorem²¹ we obtain:

$$\mathbb{P}(\text{default}_i) = \mathbb{P}(\phi_i \geq 1) \simeq \mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1). \quad (21)$$

for all $i \in \{1, \dots, N\}$.

The systemic default probability, $\mathbb{P}(\text{default})$, depends on the process governing the leverage $\phi(t)$ that, in turn, depends on the process governing the value of the portfolio of external assets $s(t)$. Because the relation between $\phi(t)$ and $s(t)$ is monotonic, there is a mapping between the passage times of the two processes and the systemic default probability can be computed more easily in terms of s (see 2.6):

$$\mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1) \equiv \mathbb{P}(s \leq s^-) \quad (22)$$

$$\text{with } \begin{cases} s^+ &= \frac{l(\beta\varepsilon + R - w\varepsilon)}{\varepsilon(\beta\varepsilon + R)} & \text{safe boundary} \\ s^- &= \frac{l(\beta + R - w)}{\beta + R} & \text{default boundary} . \end{cases}$$

PROPOSITION 1. *Consider the default conditions in Eq.(22) and assume that external assets dynamics follows a time-homogeneous process with the same drift and same variance. If external assets are uncorrelated, $\mathbb{P}(\text{default})$, can be expressed as*

$$\mathbb{P}(\text{default}) = \left(\int_{s_0}^{s^+} ds \psi(x) \right) / \left(\int_{s^-}^{s^+} dx \psi(x) \right), \text{ where } \psi(x) = \exp \left(\int_0^x -\frac{2\hat{\mu}}{\hat{\sigma}^2} ds \right). \quad (23)$$

and has the following closed form solution:

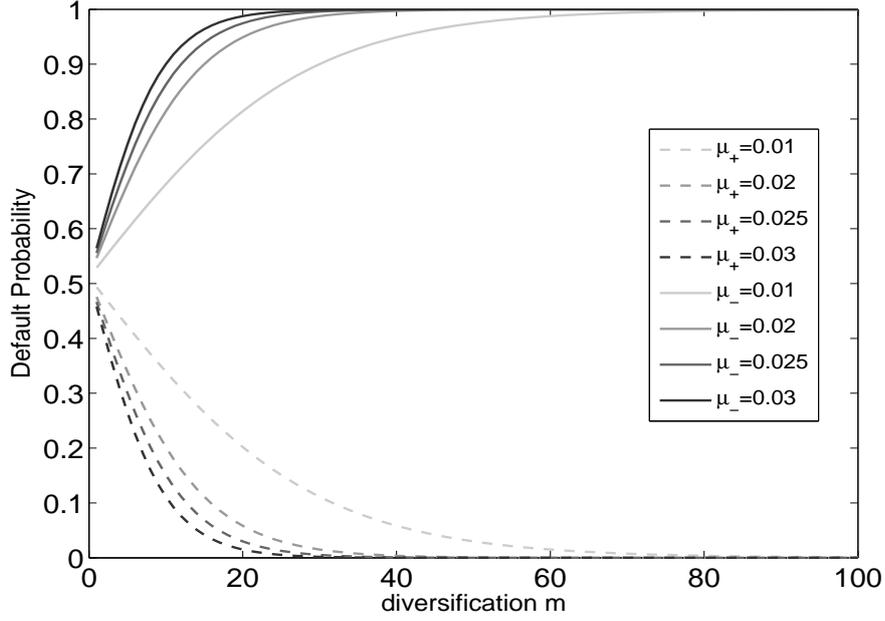
$$\mathbb{P}(\text{default}) = \left(\exp \left[-\frac{2\hat{\mu}s_0}{\hat{\sigma}^2} \right] - \exp \left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right) / \left(\exp \left[-\frac{2\hat{\mu}s^-}{\hat{\sigma}^2} \right] - \exp \left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right).$$

Proof: see 2.6.

²¹In a nutshell, the theorem states that continuous functions preserve limits even if their arguments are sequences of random variables. See Billingsley (1968).

Eq. (32) is the probability that s , initially at an arbitrary level $s(0) := s_0 \in (s^-, s^+)$, exits through s^- before s^+ after time $t > 0$. This can be related to the problem of first exit time through a particular end of the interval (s^-, s^+) (Gardiner, 1985).

Figure 3: Default probability for different levels of risk diversification.



Default probability q given a downtrend (solid lines). Default probability g given an uptrend (dashed lines). q increases with diversification. Instead, g decreases with diversification. The elasticity of q and g with respect to m depends on the magnitude of the trend. $|\mu_+| = |\mu_-| \in \{0.01, 0.02, 0.025, 0.03\}$, $l=0.4$, $s_0 = (s^+ - s^-)/2$, $\sigma^2 = 0.5$, $r_f = 0.01$, $\epsilon = 0.1$, $\beta = 0.2$, $w = 1$.

Now we need the following definition.

DEFINITION 1. *The conditional systemic default probability conditional to a given market trend is :*

$$\begin{cases} q := \mathbb{P}(\text{default} \mid \mu^-) & \text{def. prob. in the case of a downtrend ,} \\ g := \mathbb{P}(\text{default} \mid \mu^+) & \text{def. prob. in the case of an uptrend} \end{cases}$$

From Proposition 1 and Definition 1 we can now derive the following Proposition:

PROPOSITION 2. *The conditional systemic default probability, conditional to a given market trend has the following closed form solutions:*

$$\begin{cases} q = \left(\exp \left[-\frac{(2\mu^-)s_0}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right) / \left(\exp \left[-\frac{(2\mu^-)s^-}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right) , \\ g = \left(\exp \left[-\frac{(2\mu^+)s_0}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) / \left(\exp \left[-\frac{(2\mu^+)s^-}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) . \end{cases}$$

Proof: 2.6.

In such setting, the first passage time of this process must account for the new boundary. A bank could in principle reach at time $t^i > 0$ the safe boundary ε and then only later at time $t^{ii} > t^i > 0$ reach the default boundary at 1. Notice, however, that the probability of such an event is very low and this may occur only in a scholastic example when: 1) the variance σ is much larger than the trend μ ; 2) the two barriers s^+ and s^- are very close to each other; and 3) the initial value of external assets s_0 is very close to the boundaries.²²

Diversification effects on Default Probability The analysis of q and g reveals that, in an idealized world without transaction costs and infinite population size of external assets (i.e., $M \rightarrow \infty$), at increasing levels of risk diversification (i.e., $m \rightarrow M$), the default probability exhibits a bifurcated behavior when trends are persistent (i.e., approximately constant during a given period Δt) See Figure 3. We can formalize the result in the following Proposition.

PROPOSITION 3. *Consider a debt-financed portfolio subject to a given default threshold as in the model described so far. Then, the default probability decreases with risk diversification in case of asset prices uptrend and increases with diversification in case of asset prices downtrend.*

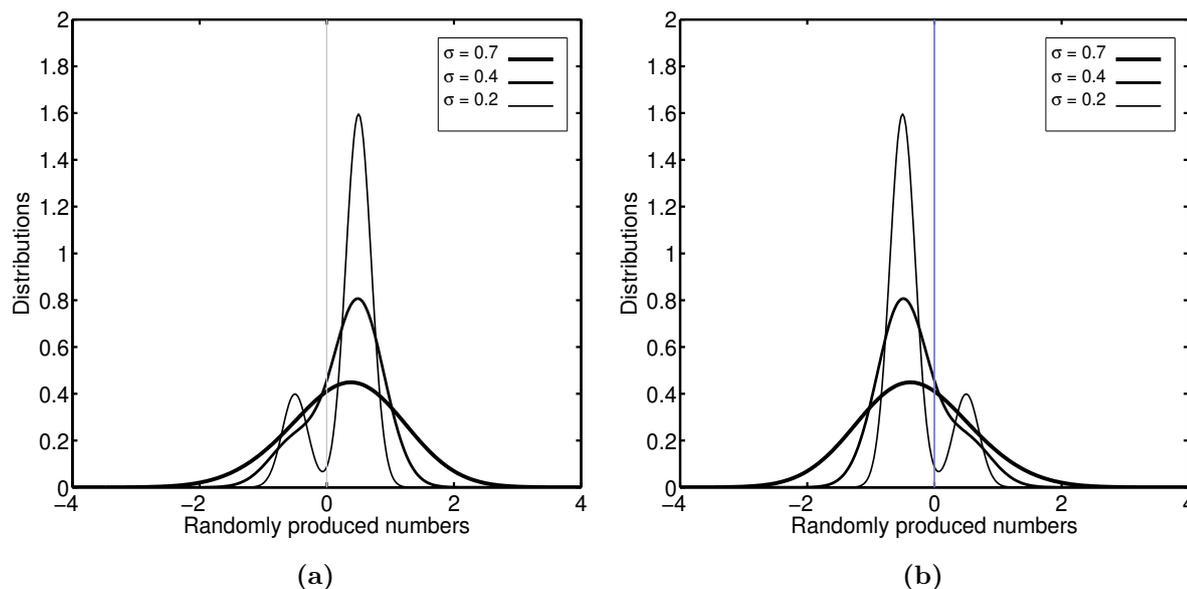
Proof: see 2.6.

The intuition behind the polarization of the probability to “survive” and the probability to “fail” is simple, but its implications are profound. In brief, the diversification of idiosyn-

²²See also 2.7 for a numerical validation of the approximation.

cratic risks reduces the volatility of the portfolio. The lower volatility increases the likelihood of the portfolio to follow an underlying economic trend. Therefore:

Figure 4: Distribution of portfolio returns for different levels of volatility and different prob. of downtrend.



Comparison between the probability density functions of three portfolios with returns following a bimodal distribution with the same expected value but decreasing volatility. Thick black curve ($\sigma = 0.7$), medium black curve ($\sigma = 0.4$), thin black curve ($\sigma = 0.2$). **(a)** Each curve represents the pdf of a portfolio with returns distributed as $Y1 \sim N(-0.5, \sigma)$ with probability $p = 0.2$ and as $Y2 \sim N(0.5, \sigma)$ with probability $1 - p = 0.8$. **(b)** Each curve represents the pdf of a portfolio with returns distributed as $Y1 \sim N(0.5, \sigma)$ with probability $p = 0.8$ and as $Y2 \sim N(-0.5, \sigma)$ with probability $1 - p = 0.2$.

- in uptrend periods, diversification is beneficial because it reduces the downside risk and highlights the positive trend; thus, the default probability decreases;
- in downtrend periods, diversification is detrimental because it reduces the upside potential and highlights the negative trend; therefore, the default probability increases.

Figure 4 explains this intuition by showing how the pdf of portfolio returns is influenced by the downtrend probability and by the level of diversification. As one may observe, for

increasing diversification, viz., lower volatility, the pdf changes shape by moving from the thick black curve ($\sigma = 0.7$), to the medium black curve ($\sigma = 0.4$), and finally to the thin black curve ($\sigma = 0.2$). In Figure 4 a) the probability of a positive trend is greater than the probability of a negative trend, $p = 0.2$.

Therefore, diversification is desirable because it reduces the volatility and, by doing so, it shifts to the right the density of the distribution of portfolio returns. Instead, In Figure 4 b) the probability of a negative trend is greater than the probability of a positive trend, $p = 0.8$. In this case, diversification is undesirable because, by reducing the volatility, it shifts to the left the density of the distribution.

2.3.3 The effect of network on default probability

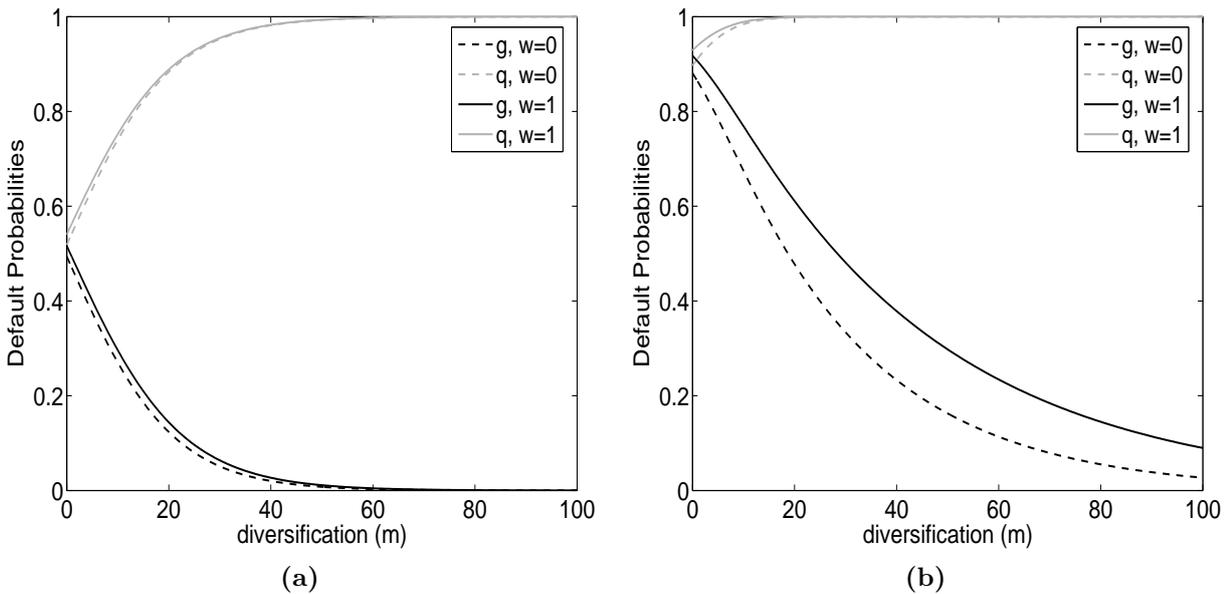
In order to illustrate the effect of the interbank network, we compare the case in which banks are connected in an interbank market ($w = 1$) with the case in which banks are not exposed to the interbank market ($w = 0$). However, in order to make the balance sheets comparable in the two cases in terms of leverage, we replace the interbank claims with other independent assets of equivalent face value. These latter assets are independent from the external assets and are constant in value. In these cases, we compute the closed form solutions for the default probabilities q and g in the two scenarios.

In Figure 5 we plot the probabilities g and q for increasing levels of diversification. Dashed lines refer to the case of isolated banks, while solid lines indicate the case of interbank exposures. As we can see, the default probability with interbank exposures is always higher than the corresponding default probability in the case of isolated banks. This holds true both for the scenario of uptrend and downtrend. Moreover, in Figure 6 we plot the ratio between the default probability conditional to the downtrend g , $\frac{g_{w=1}}{g_{w=0}}$, in the case with and without interbank exposures, for different portfolio leverage values. In the two examples shown, the ratio between the two curves increases either because of a higher level of diversification m (along x-axis) or because of the balance sheet structure (e.g. the values of s_0 and l , see bold line vs dashed line in Figure 6). On one hand, with higher level of leverage ϕ (dashed line),

the average effect of the network externality on banks becomes stronger because, whenever an agent suffers a relatively larger shock, the impact is reinforced by the network structure and the initial default of one agent causes a big loss to the counterparties that are already fragile ($\phi \approx 1$); on the other hand, since a greater level of diversification implies a lower probability of default in case of downtrend, the results indicate that the impact of network effect is larger for a lower level default probability of the bank.

The reason behind these results is the fact that increases in leverage of the counterparties k of the bank i cause an increase in the leverage of bank i through the market value of the interbank claims. This effect is proportional to the exposure of the bank to the interbank market. This is stated in the following Proposition.

Figure 5: Probabilities of default with and without interbank.



Comparison between def. probabilities g and q with interbank exposures (bold lines) and without interbank exposures among banks (dashed lines). Eliminating interbank spillovers from the setting decreases in both scenarios the probabilities of default q and g . We test for different values of s_0 . Parameters: **(a)** $l = 0.9$, $s_0 = 4$, $p = 0.4$, $\beta = 0.5$, $r_f = .001$, $\epsilon = 0.1$, $w \in \{0, 1\}$; **(b)** $l = 1.5$, $s_0 = 1.5$, $p = 0.4$, $\beta = 0.5$, $r_f = .001$, $\epsilon = 0.1$, $w \in \{0, 1\}$.

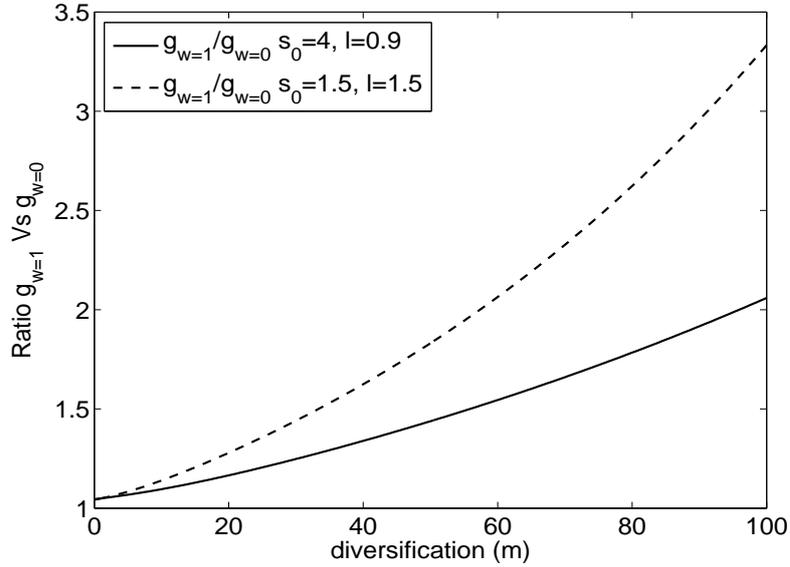
PROPOSITION 4. *Assume homogeneity in the banks' capital structure and the presence of a*

network of interbank exposures among banks. For a given default threshold, *ceteris paribus*, the leverage of an individual bank, measured in terms of its leverage ϕ , increases non-linearly with the level of its borrowers leverage and proportionally to the level of its interbank exposures.

Proof: 2.6.

We conclude by saying that, for a given change in counterparties' leverage ϕ_k , an increment in both the interbank network density ($w \rightarrow 1$) and in the level of diversification ($m \rightarrow M$) amplifies the variations of individual banks' leverage and their default probability.

Figure 6: Ratio between probabilities of default g calculated with and without interbank.



Ratio, $\frac{g_{w=1}}{g_{w=0}}$, between the default probability g (conditional to uptrend) in the case with and without interbank exposures. The following parameter sets are considered: solid lines: $l = 0.9$, $s_0 = 4$, $p = 0.4$, $\beta = 0.5$, $r_f = .001$, $\epsilon = 0.1$, $w \in \{0, 1\}$; dashed line: $l = 1.5$, $s_0 = 1.5$, $p = 0.4$, $\beta = 0.5$, $r_f = .001$, $\epsilon = 0.1$, $w \in \{0, 1\}$.

2.3.4 Bank Utility Function

In this section we formalize the bank utility maximization problem with respect to the number m of external assets held in the equally weighted portfolio described in Section 2.3.1.

The bank's payoff from investing in external assets is a random variable Π_m that depends on the number m of external assets in portfolio and on their values. It takes the value π in the set $\Omega_\pi = [\pi^-, \dots, \pi^+]$, where:

$$\begin{cases} \pi^- := s^- - s_0 & \text{max attainable loss,} \\ \pi^+ := s^+ - s_0 & \text{max attainable profit.} \end{cases}$$

More specifically, given m mutually exclusive choices (i.e., the bank portfolio can be composed of 1, 2, ... external assets) and their corresponding random return Π_1, Π_2, \dots , with distribution function $F_1(\pi), F_2(\pi), \dots$, preferences that satisfy the von Neumann-Morgenstern axioms imply the existence of a measurable, continuous utility function $U(\pi)$ such that Π_1 is preferred to Π_2 if and only if $\mathbb{E}U(\Pi_1) > \mathbb{E}U(\Pi_2)$. We assume that banks are mean-variance (MV) decision makers, such that the utility function $\mathbb{E}U(\Pi_m)$ may be written as a smooth function $V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m))$ ²³ of the mean $\mathbb{E}(\Pi_m)$ and the variance $\sigma^2(\Pi_m)$ of Π_m or

$$V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) := \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - (\lambda\sigma^2(\Pi_m))/2$$

such that Π_1 is preferred to Π_2 if and only if ²⁴

$$V(\mathbb{E}(\Pi_1), \sigma^2(\Pi_1)) > V(\mathbb{E}(\Pi_2), \sigma^2(\Pi_2)).$$

Then, the maximization problem is as follows:

$$\max_m \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2} \quad (24)$$

²³To describe V as smooth, it simply means that V is a twice differentiable function of the parameters $\mathbb{E}(\Pi_m)$ and $\sigma^2(\Pi_m)$.

²⁴Only the first two moments are relevant for the decision maker; thus, the expected utility can be written as a function in terms of the expected return (increasing) and the variance (decreasing) only, with $\partial V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) / \partial \mathbb{E}(\Pi_m) > 0$ and $\partial V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) / \partial \sigma^2(\Pi_m) < 0$.

$$\text{s.t.:} \begin{cases} 1 \leq m \leq M \\ l > 0 \\ s^- < s_0 < s^+ \\ p\mu_s^- + (1-p)\mu_s^+ > 0 \end{cases}$$

$$\text{with} \begin{cases} \mathbb{E}(\Pi_m) = p[q\pi^- + (1-q)\pi^+] + (1-p)[g\pi^- + (1-g)\pi^+] \\ \sigma^2(\Pi_m) = p \left[q(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)(\pi^+ - \mathbb{E}(\Pi_m))^2 \right] \\ \quad + (1-p) \left[g(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)(\pi^+ - \mathbb{E}(\Pi_m))^2 \right]. \end{cases}$$

Notice that Eq. (24) is a *static* non-linear optimization problem with respect to m , with inequality constraints.²⁵

The first constraint means that m can take only positive values between 1 and M . The second constraint requires banks to be debt-financed. The third constraint requires that banks are not yet in default when implementing their asset allocation, i.e., the initial portfolio value must lie between the lower default boundary and the upper safe boundary. The last constraint represents the “economic growth condition”. That is, the expected economic trend of the real economy-related assets has to be positive. Since by definition, $|\mu_s^-| = |\mu_s^+| = \mu_s$, this condition is equivalent to impose: 1) a lower bound on the trend, namely $\mu_s > 0$, and 2) an upper bound to the downtrend probability, namely $p \in \Omega_p := [0, \frac{1}{2}]$. Then, given the above constraints, at time t banks randomly select (and fix) the number m of external assets to hold in their portfolio in order to minimize their default probability. This, in turn, maximizes their expected utility.

Finally, in the next section we will show that the expected utility in Eq.(24) is lower when banks hold external assets with higher initial values, s_0 . This counter-intuitive property is explained by the non-linear payoff function formalized at the beginning of this section. A

²⁵Although the realized state of the world will be either an uptrend or a downtrend, banks implement their portfolio diversification strategy before the realization of the trend. Under this condition of uncertainty, from the perspective of portfolio selection problem, returns display a mixture distribution (expressed by the convex combination of two normal distributions weighted by p and $1-p$) as shown in Eq. (16). Coherently, also the expected utility in Eq.(24) follows a similar mixture property with expected profit and variance expressed by the convex combination of two alternative payoffs weighted by p and $1-p$.

higher initial price s_0 increases maximal losses and decreases maximal profits. For example, if the maximum (minimum) attainable value by the external assets is 100 (0) then, when $s_0 = 90$ the maximum profit (loss) is 10 (90). Instead, if the $s_0 = 10$, the maximum profit (loss) is 90 (10). This effect combines non-linearly into Eq.(24) and leads to a higher (lower) bank expected utility for lower (higher) levels of s_0 . This mechanism resembles profit/loss dynamics from investments in a boom-bust cycle. Banks buy assets at high (low) price in a boom (bust) period. Correspondingly, in absolute terms, their maximum attainable profit is lower (higher) and their maximum attainable loss is higher (lower).

2.3.5 Solution of the Bank Max Problem

Notice that, the exposure of banks to external assets is both direct through their own portfolio and indirect, through their interbank claims on counterparties who are also exposed to similar portfolios of external assets. In the following, banks compute the optimal level of diversification taking this indirect effect into account. In other words, because the expression for the probability of systemic default takes into account the effect of contagion through the interbank market, banks internalize the negative externality of contagion in their utility function.

Banks face the ex-ante uncertainty on which economic trend will occur and they know that they will not be able to divest from their external assets. As shown in the previous sections, maximal risk diversification minimizes the default probability if the economic trend happens to be positive, because it reduces the downside risk. In contrast, a minimal risk diversification minimizes the default probability when the economic trend happens to be negative. The intuition could then suggest that the optimal level of risk diversification is always interior and depends on the probability of the market trend to be positive or negative. However, this intuition suffers from the logical fallacy that maximizing a convex combination of functions is equivalent to take the convex combination of the maxima, which is not correct in general.

We show that optimal risk diversification is interior under certain conditions, but is not

interior in general. In particular, we find that:

- banks with small enough share of asset invested in the external assets ($s_0 < s^*$), maximize their expected utility at extremal levels of diversification, i.e., $m^* = M$;
- banks with large enough share of asset invested in the external assets ($s_0 \geq s^*$), maximize their expected utility at intermediate levels of diversification, i.e., $1 < m^* < M$.

Formally, this is stated in the following Proposition.

PROPOSITION 5. *Given the probability interval $\Omega_p := [0, \frac{1}{2})$ and under the existence of two possible states of the world where asset returns either follow a normally distributed downtrend (or uptrend) with mean equal to μ^- (or μ^+) and variance σ , the banks' expected utility maximization problem, as formalized in Eq. (24), leads to the following alternative solutions:*

1. *If $s_0 < s^*$, the expected utility $\mathbb{E}U(\Pi_m)$ is maximized when $m^* = M$. Then, $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$, for all $m < M$.*
2. *if $s_0 \geq s^*$, there exists a subinterval $\Omega_{p^*} \subset \Omega_p$ s.t., to each $p^* \in \Omega_{p^*}$ corresponds an optimal level of diversification m^* in the open ball $B(\frac{1+M}{2}, r) = \{m^* \in \mathbb{R} \mid d(m^*, \frac{1+M}{2}) < r\}$ with center $\frac{1+M}{2}$ and radius $r \in [0, \alpha]$ where $\alpha = f(q, g)$. Then, $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$, for all $m \notin B(\frac{1+M}{2}, r)$.*

Proof: 2.6.

Proposition 5 II. states that the optimal diversification can be an interior solution.²⁶ Namely, when banks maximize their MV utility they may choose an intermediate level of diversification, viz., $1 < m^* < M$. m^* is the unique optimal solution and its level depends on the market size and on the likelihood of incurring in a negative or positive trend.

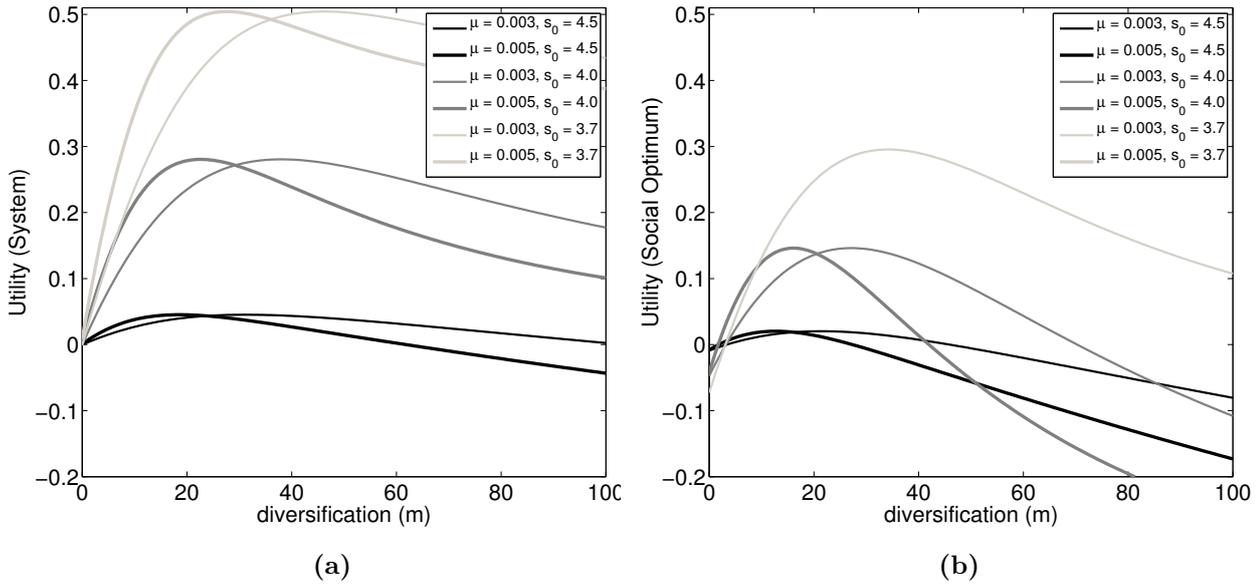
In the words of Haldane (2009), we show that diversification is a *double-edged* strategy. Values of $m \gtrless m^*$ are second-best choices. Precisely, by increasing m to approach m^* from below, banks increase their utility. However, by increasing m beyond m^* , banks decrease

²⁶As a robustness check, we tested the validity of the Proposition for different values of p . Results are robust also for low values of the probability of downtrend.

their utility. In summary, the MV utility exhibits inverse U-shaped non-monotonic behavior with respect to m . These results hold under the market structure described in the previous sections.

Briefly, banks are fully rational agents with incomplete information about the future state of the world. There are no transaction costs, or market asymmetries. Market returns exhibit a bimodal distribution. There are negative externality on the interbank arising from the fact that banks' interbank assets are negatively affected when a counterparty's leverage increases. In our model, this externality is internalized by each bank since their utility function is computed by using a default probability model that accounts for the contagion from counterparties.

Figure 7: Expected MV Utility for different levels of risk diversification.

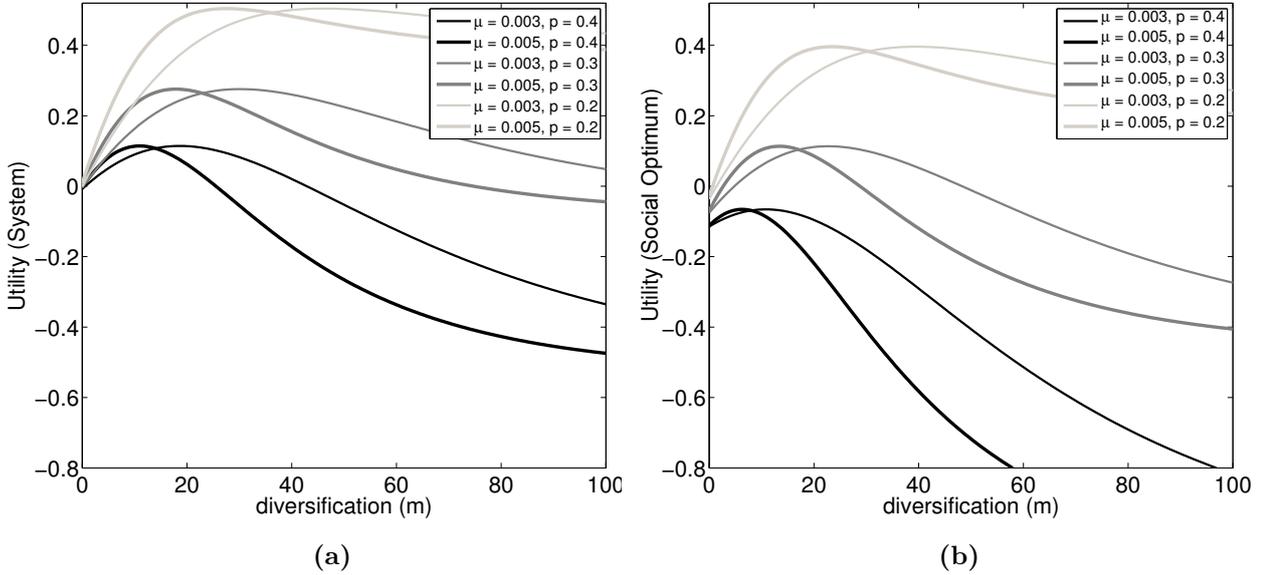


Fixed downtrend probability, $p = 0.4$. The figure compares the expected MV utility of the banking system vs. the expected MV utility in the social optimum case. The curves represent different initial asset values, s_0 , and different drifts, μ . **(a)** Exp. MV Utility of the banking system $\mathbb{E}U(\Pi_m)$. **(b)** Exp. MV Utility in the social optimum case $\mathbb{E}U_r(\Pi_m)$. Parameters: $\sigma^2 = 0.5$, $r_f = 0.001$, $\varepsilon = 0.1$, $\lambda = 0.1$, $\beta = 0.2$, $l = 0.5$, $k = 2$, $m \in \{1, \dots, 100\}$, $s_0 \in \{3.7, 4, 4.5\}$, $w = 1$, $|\mu^+| = |\mu^-| \in \{0.003, 0.005\}$.

For a fixed probability p , Figure 7 a) shows how the utility changes for different levels

of diversification m , different magnitudes of the trend and different initial asset values, s_0 . Instead, for a fixed initial asset value s_0 , Figure 8 a) shows how the utility changes for different levels of diversification m , different magnitudes of the trend and different probability of downtrend, p . Notice that, m enters into the maximization problem via Eq. (15). In particular, the portfolio volatility decreases with m . Therefore, an outward movement along the x -axis in Figures 7–8, i.e., increasing m , is equivalent to a market condition where the volatility of the assets is low, $\sigma = \sigma_{low}$.

Figure 8: Expected MV Utility for different levels of risk diversification.



Fixed initial asset value, $s_0 = 3.69$. The figure compares the expected MV utility of the banking system vs. the expected MV utility in the social optimum. The curves represent different probabilities of downtrend, p and different drifts, μ . **(a)** Exp. MV Utility of the banking system $\mathbb{E}U(\Pi_m)$. **(b)** Exp. MV Utility in the social optimum $\mathbb{E}U_r(\Pi_m)$. Parameters: $\sigma^2 = 0.5$, $r_f = 0.001$, $\varepsilon = 0.1$, $\lambda = 0.1$, $\beta = 0.2$, $l = 0.5$, $k = 2$, $m \in \{1, \dots, 100\}$, $w = 1$, $p \in \{0.2, 0.3, 0.4\}$, $|\mu^+| = |\mu^-| \in \{0.003, 0.005\}$.

Conversely, an inward movement along the x -axis in Figures 7–8, i.e., decreasing m , is equivalent to a market condition where the volatility of the assets is high, $\sigma = \sigma_{high}$. To conclude, one might observe that if banks are already in m^* , an abrupt increases in the volatility of the assets (equivalent to an inward movement from m^*) decreases the utility of

the banks.²⁷ In an optimization problem similar to our own, but with endogenous equilibrium asset pricing, Danielsson and Zigrand (2008) show that an increase in the volatility of both assets and portfolios can be generated by imposing strict risk-sensitive constraints of the VaR type.

2.3.6 Conditional Systemic Default Probability

In this section, we analyze the probability of a systemic default conditional to the default of any one bank j , defined as $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1)$. In particular, two basic properties hold. If banks do not have interbank exposures and are not diversified on their external assets, the conditional systemic default probability tends to zero for large number of banks, i.e. banks never default together.

On the contrary, when banks do have interbank exposure and are fully diversified in external assets, the *conditional* systemic default probability is identically one, i.e. banks always default together. Formally:

PROPOSITION 6. *Denote the probability of a systemic event conditional to the default of bank j by $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1)$. For any value of interbank exposure $1 \geq w \geq 0$ and some $j \in \{1, \dots, N\}$, it holds:*

If $m = 1$: $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1) \rightarrow 0$ for $n \rightarrow \infty$;

If $m = M$: $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1) = 1$ for $n \rightarrow \infty$.

Proof: 2.6.

In 2.8, we perform a set of Monte Carlo simulations in order to verify the above Proposition in the context of n banks and to investigate how the conditional probability of systemic default varies for intermediate levels of diversification m when banks hold overlapping portfolios. The conditional probability of simultaneous defaults will play an important role in

²⁷This result could surprise as a possible explanation for the empirical findings that solvency condition across financial institutions, in the recent US 2007-08 crises, has been driven by an increase in the volatility of the firm's assets, Atkeson et al. (2013).

the tension between individual banks and socially optimal diversification that is analyzed in the following section.

2.4 Private Incentives vs. Social Optimum

In the following, we assume that there are social costs associated to the default of one or more banks due to negative externality to the real economy that limited-liability banks commonly do not account for. We also assume that it is more costly to restore the functioning of the market in case of simultaneous defaults than in the case of isolated defaults. We intentionally leave the definition of social costs general as it may depend on the characteristics of the financial system under analysis. Consider a sophisticated bank that, differently from the other banks, internalizes the social costs. For simplicity we define such case as “social optimum”. We want to formulate the utility maximization problem in the “social optimum” and compare it to the utility maximization problem of individual banks in Eq. (24)²⁸.

Let K be the number of simultaneously defaulting banks. We make the following assumptions:

ASSUMPTION 1. *The total loss to be accounted in the case of social optimum in downturn periods is a monotonically increasing function $f(k, \pi^-) := k\pi^-$ of: (i) the expected number k of bank crashes given a collapse of at least one bank $\mathbb{E}(K|K \geq 1) = k$, (ii) the magnitude of the loss π^- .*

2.4.1 Social Optimum Utility Function

Therefore, the social optimum’s utility maximization problem is as follows,

$$\max_m \mathbb{E}U_r(\Pi_m) = \mathbb{E}_r(\Pi_m) - \frac{\lambda\sigma_r^2(\Pi_m)}{2} \quad (25)$$

²⁸In the context of our model, we analyze the misalignments between private incentives and social optimal level of diversification, but we do not formalize the problem of a social planner who intends to maximize social welfare by choosing the extent of diversification in banks’ exposures to external assets. Rather, we assess social optimality by comparing the optimal level of diversification chosen by single banks with that of a bank internalizing the negative externalities caused by banks’ failures in the system.

$$\text{s.t.:} \begin{cases} 1 \leq m \leq M \\ l > 0 \\ s^- < s_0 < s^+ \\ p\mu_s^- + (1-p)\mu_s^+ > 0 \\ k > 1 \end{cases}$$

$$\text{with} \begin{cases} \mathbb{E}_r(\Pi_m) = p[qk\pi^- + (1-q)\pi^+] + (1-p)[g\pi^- + (1-g)\pi^+] \\ \sigma_r^2(\Pi_m) = p[q(k\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)(\pi^+ - \mathbb{E}(\Pi_m))^2] \\ \quad + (1-p)[g(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)(\pi^+ - \mathbb{E}(\Pi_m))^2] . \end{cases}$$

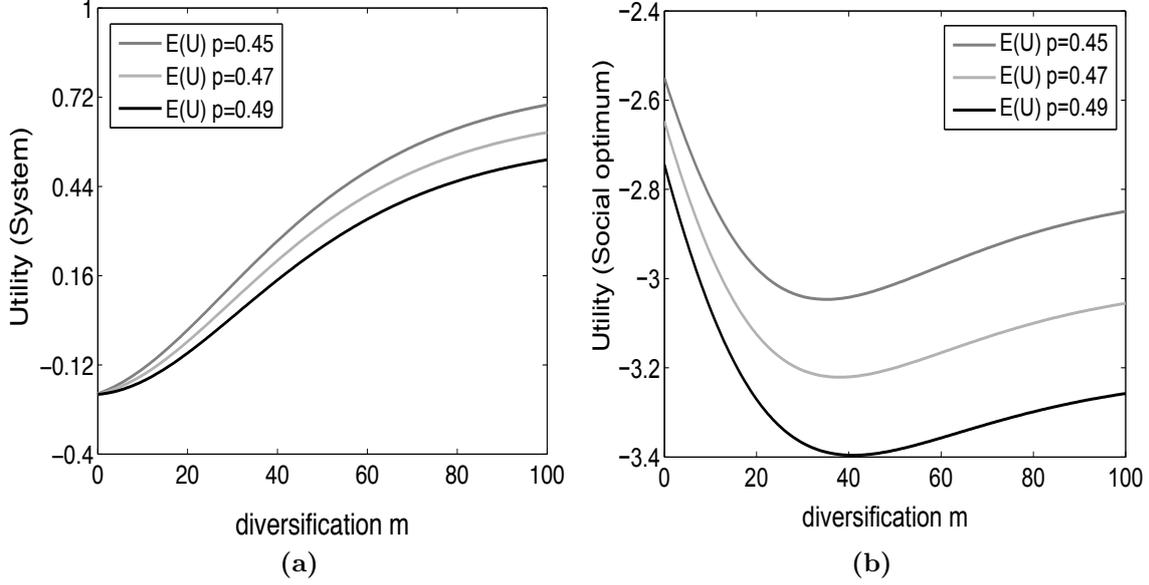
Notice that, with respect to the optimization in Eq. (24), the social optimum is subject to the additional constraint $k > 1$ that amplifies both the expected loss and its variance. In summary, we compare the general solution of the maximization problem in Eq. (24) with that in Eq. (25).

2.4.2 The tension between individual and social optimum's diversification

The results shown in the previous sections imply the emergence of a tension between the incentives of the individual banks and the social optimum, a finding that is important also in terms of policy implications. Consider the following (not implausible) scenario in which the downtrend is less probable than the uptrend and yet its probability is not negligible. Then, banks will prefer a maximal diversification. As shown in 2.8, in so doing, they will hold completely correlated portfolios and they will tend to default altogether. This implies a very high social cost in the social optimum, since banks would have instead very low diversification in order to keep defaults less correlated. The tension between private and public incentive is extreme in this case. In the following Proposition we show that the optimal level of diversification for the social optimum, m^r , is lower than the level that is desirable for individual banks, m^* .

PROPOSITION 7. *If banks do not internalize the impact of social costs in the case of simulta-*

Figure 9: Tension between individual bank and social optimum



Comparison between expected utility function of **(a)** individual banks **(b)** and social optimum. Parameters: $p \in \{0.45, 0.47, 0.49\}$, $\sigma^2 = 0.5$, $r_f = 0.001$, $\epsilon = 0.1$, $\lambda = 0.1$, $\beta = 0.2$, $w = 1$, $l = 0.5$, $k = 5$, $m \in \{1, \dots, 100\}$ $s_0 = 1.5$, $|\mu^+| = |\mu^-| = 0.003$.

neous defaults, in equilibrium the banking system is over-diversified on external assets w.r.t. to the level of diversification that is socially desirable:

$$m^* \geq m^r \quad (26)$$

and

$$m^r < \infty \quad (27)$$

$\forall s_0 \in (s^-, s^+)$.

Proof: 2.6.

To illustrate the result in Proposition 7 we compare Figure 7 a) with Figure 7 b) and Figure 8 a) with Figure 8 b). We can observe that the tension between individual and social incentives leads to a banking system that maximizes its expected utility at a level of

diversification higher than the one maximizing the expected utility in the social optimum, viz. $m^* \geq m^r$. Moreover, there is a range of parameters where the tension is maximal because $m^* = M$ and $m^r = 1$. This is illustrated in Figure 9 and in the following numerical result. See Appendix C.

REMARK 3. *If the external asset price is small enough, $s_0 < s^*$ (see Proposition 5.I), there exists a range of p , such that $\{p \in \mathbb{R} \mid d(p, 0.5) < r\}$ with $r > 0$, in which the expected utility of the banking system is maximized for the maximum level of diversification while the expected utility in the social optimum is maximized for the minimum level of diversification.*

$$m^* = M > m^r = 1. \tag{28}$$

See Appendix C.

2.5 Concluding Remarks

This paper develops a new modeling framework in order to investigate the probability of default in a system of banks holding claims on each others as well as overlapping portfolios due to bank diversification in a market with a finite number of external assets.

While our framework, uniquely combining together network and portfolio theories, allows for the investigation of several cross-disciplinary questions, we focus on the *core* of tightly knit banks, i.e., $w = 1$. In this setting, we study how the probability of a systemic default depends on the level of diversification in external assets.

To incorporate the network effect, we measure the performance of portfolio diversification by its impact to banks' default probabilities and expected utility, rather than by its impact on the asset portfolio volatility. By doing that, we extend the literature focusing on the hedging efficiency in the context of portfolio management alone, e.g., Markowitz (1952), Johnson and Shannon (1974). Indeed, the network of interbank claims exposes each individual bank to shocks on those external assets held by its counterparties. The network amplifies the effect of the negative trend, when it occurs, and the impact of shocks when banks have largely

overlapping portfolios due to extensive diversification on external assets. In general terms, our paper (implicitly) argues that firms' optimal hedging choice should incorporate firms' balance sheet structure as a whole, such as counterparty exposures and leverage ratio.

As a second contribution, this paper offers a complementary approach with respect to the strand of works (on the valuation of distressed assets) building on Eisenberg and Noe (2001). In those works the default process is considered only at the maturity of the interbank contracts. Differently, in this paper the values of the interbank claims are dependent among each others and evolve over time as a system of stochastic processes. By so doing, the contagion propagated through the network is internalized by the banks even before the maturity of the contracts.

A third contribution consists of investigating the optimal diversification strategy of banks in the face of opposite and persistent economic trends that are unknown to banks. Maximal risk diversification minimizes the default probability if the economic trend happens to be positive, because it reduces the downside risk. In contrast, a minimal risk diversification minimizes the default probability when the economic trend happens to be negative because gives the chance to "gamble for resurrection". If banks, for various constraints, are not able to change diversification strategy during a certain time, then we could intuitively think that the optimal level of risk diversification is always interior because it must be a convex combination of the two optima. The exact level of interior optimal diversification should depend on the probability of the market trend to be positive or negative. However, this intuition suffers from a logical fallacy. In fact, maximizing a convex combination of functions is *not* equivalent to take the convex combination of the maxima. Accordingly, we show that optimal risk diversification is interior under certain conditions, but, in general, it is *not* interior. We find that banks' optimal diversification is extremal when the exposure to the external assets is not too large and it is interior otherwise.

Notice that banks do internalize the negative externality resulting from banks holding claims on counterparties' liabilities, the value of which is affected by counterparties leverage. Indeed, the default probability, used by banks when computing the optimal diversification,

takes into account contagion.

As a fourth contribution, this paper offers an analysis of the tension arising between banks' incentives and social optimum. For the largest range of the parameters, bank's optimal choice leads to full (or very high) diversification across the external assets. However, this also leads to banks having highly correlated portfolios and suffering simultaneous defaults. The later may engender high social costs for the real economy. When those social costs are internalized, the socially desirable level of diversification is *lower* than the level chosen by the banking system.

Our work could be extended in a number of ways. In particular, one could study both the effect of diversification across counterparties and external assets. Notice that, in order to better isolate the effectiveness of diversification in mitigating idiosyncratic risks to which banks are exposed via external assets holding, in our setting the interbank diversification is fixed and homogeneous. However, one could introduce some heterogeneity in the balance sheet structure and in the portfolio holdings of the banks and answer the question whether external diversification and interbank diversification are substitutes or complements.

Overall, an important point stemming from our analysis lies in the recognition that social optimum is not to target a specific diversification level of risk but rather to manage the trade-off between the social losses from defaults (because of excessive risk spreading in economic downturn) and the social costs of avoiding defaults (because of excessive risk diversification in economic booms).

2.6 Proofs

Remark 1. If f or $\phi_i \rightarrow 1$ the marked-to-market value of interbank liabilities is such that $\hat{\eta}_i > 0$, the implicit recovery rate is:

$$\delta = \begin{cases} 1 & \text{in case of no default} \\ \frac{1+r_f}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

Proof. The recovery rate is the proportion of face value that is recovered through bankruptcy

procedures in the event of a default. Therefore, the general formula for the discounted recovery rate $\hat{\delta}$ is

$$\begin{aligned}\hat{\delta} &= \frac{l_i}{(1+r_f+\beta\phi_i)} \left(\frac{1}{l_i}\right) \\ &= \frac{1}{(1+r_f+\beta\phi_i)}.\end{aligned}\tag{29}$$

That is

$$\hat{\delta} = \begin{cases} \frac{1}{1+r_f} & \text{in case of no default} \\ \frac{1}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

The actual recovery rate at the maturity of the debt is $\delta = \hat{\delta}(1+r_f)$. That is

$$\delta = \begin{cases} 1 & \text{in case of no default} \\ \frac{1+r_f}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

□

Remark 2. Bank's leverage defined as in Eq. 10 is a second order polynomial equation in Φ :

$$\Phi \mathbf{I} \beta \Phi + \Phi \mathbf{I} \mathbf{R} - \mathbf{L}(\mathbf{W}\mathbf{H})^{-1} \beta \Phi + \Phi = \mathbf{L}(\mathbf{W}\mathbf{H})^{-1} \mathbf{R}\tag{30}$$

where $\mathbf{H} := \text{diag}(\eta_1, \eta_2, \dots, \eta_M)$; $\mathbf{I} := \mathbf{Z}\mathbf{V}(\mathbf{W}\mathbf{H})^{-1}$ and $\Phi := \text{diag}(\phi_1, \phi_2, \dots, \phi_N)$; $\mathbf{L} := \text{diag}(l_1, l_2, \dots, l_N)$; $\mathbf{V} := \text{diag}(\nu_1, \nu_2, \dots, \nu_M)$; $\mathbf{R} := \text{diag}(R, R, \dots, R)$ with $R = 1 + r_f$; $\mathbf{W} := [w_{ik}]_{N \times N}$; $\mathbf{Z} := [z_{ij}]_{N \times M}$.

Proof. The leverage at banking system level can be derived from Eq. (10) and rewritten as:

$$l_i = \phi_i \times \left(\sum_j z_{ij} \nu_j + \sum_k w_{ik} \eta_k / (1 + r_f + \beta \phi_k) \right).\tag{31}$$

In vector notation, Eq.(31) is equivalent to

$$\mathbf{L} = \Phi \times [\mathbf{ZV} + (\mathbf{R} + \beta\Phi)^{-1}\mathbf{WH}]$$

that we explicit for Φ :

$$\mathbf{L} = \Phi [\mathbf{ZV} + (\mathbf{R} + \beta\Phi)^{-1}\mathbf{WH}]$$

$$\mathbf{L} = \Phi\mathbf{ZV} + \Phi(\mathbf{R} + \beta\Phi)^{-1}\mathbf{WH}$$

$$\mathbf{L}(\mathbf{WH})^{-1} = \Phi\mathbf{ZV}(\mathbf{WH})^{-1} + \Phi(\mathbf{R} + \beta\Phi)^{-1}\mathbf{WH}(\mathbf{WH})^{-1}$$

$$\mathbf{L}(\mathbf{WH})^{-1}(\mathbf{R} + \beta\Phi) = \Phi\mathbf{ZV}(\mathbf{WH})^{-1}(\mathbf{R} + \beta\Phi) + \Phi(\mathbf{R} + \beta\Phi)^{-1}(\mathbf{R} + \beta\Phi)$$

$$\mathbf{L}(\mathbf{WH})^{-1}\mathbf{R} + \mathbf{L}(\mathbf{WH})^{-1}\beta\Phi = \Phi\mathbf{ZV}(\mathbf{WH})^{-1}(\mathbf{R} + \beta\Phi) + \Phi$$

$$\Phi\mathbf{I}\beta\Phi + \Phi\mathbf{I}\mathbf{R} - \mathbf{L}(\mathbf{WH})^{-1}\beta\Phi + \Phi = \mathbf{L}(\mathbf{WH})^{-1}\mathbf{R}$$

where $\mathbf{I} := \mathbf{ZV}(\mathbf{WH})^{-1}$. □

Proposition 1. Consider the default conditions in Eq.(22) and assume that external assets dynamics follows a time-homogeneous process with the same drift and same variance. If external assets are uncorrelated, $\mathbb{P}(\text{default})$, can be expressed as

$$\mathbb{P}(\text{default}) = \left(\int_{s_0}^{s^+} ds\psi(x) \right) / \left(\int_{s^-}^{s^+} dx\psi(x) \right), \text{ where } \psi(x) = \exp \left(\int_0^x -\frac{2\hat{\mu}}{\hat{\sigma}^2} ds \right). \quad (32)$$

and has the following closed form solution:

$$\mathbb{P}(\text{default}) = \left(\exp \left[-\frac{2\hat{\mu}s_0}{\hat{\sigma}^2} \right] - \exp \left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right) / \left(\exp \left[-\frac{2\hat{\mu}s^-}{\hat{\sigma}^2} \right] - \exp \left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right).$$

Proof. In compact form, Eq. (19) reads as $\phi = f(s, l, r_f, \beta)$. The dynamics of ϕ depends directly on the dynamics of s in Eq. (15) because both r_f and l are real constant and β is

a coefficient. Therefore, to derive the systemic default probability $\mathbb{P}(\phi \geq 1)$ in a close form, one could find, via Ito's Lemma, the dynamics of ϕ from the dynamics of s and observe whether $\phi(0) \in (\varepsilon, 1)$ exits after time $t \geq 0$ through the *upper default boundary fixed at one*. However, f is highly non linear in s , see Eq. (19). Thus, it is convenient to derive the systemic default probability directly from the dynamics of s by mapping the sample space of ϕ into the sample space of s . Since the partial derivative of f with respect to s , i.e., $\frac{\partial f}{\partial s} = -l/[s + (lw/(\beta\phi + R))]^2$, is negative for all s and for any value of r_f, l in their range of variation, the Inverse Function Theorem implies that f is invertible on \mathbb{R}^+ :

$$f^{-1}(\phi) = \frac{l(\beta\phi + R - \phi w)}{\phi(\beta\phi + R)}, \quad \text{s.t.} \quad (f^{-1})'(\phi) = \frac{1}{f'(s)}.$$

Observe that, by definition, the value of external assets cannot be negative, $s \in \mathbb{R}^+$. Given the above result, we obtain the following mapping between the values of ϕ and s :

$$\begin{cases} f(s) = 1 & \text{iff } f^{-1}(\phi) = \frac{l(\beta+R-w)}{\beta+R} := s^- \\ f(s) = \varepsilon & \text{iff } f^{-1}(\phi) = \frac{l(\beta\varepsilon+R-w\varepsilon)}{\beta\varepsilon^2+R\varepsilon} := s^+ . \end{cases}$$

Hence, the systemic default probability can be defined also with respect to s :

$$\mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1) \equiv \mathbb{P}(s \leq s^-).$$

This is the probability that s , initially at an arbitrary level $s(0) := s_0 \in (s^-, s^+)$, exits through the *lower default boundary* s^- after time $t \geq 0$. From Gardiner (1985), $\mathbb{P}(s \leq s^-)$ has the following explicit form:

$$\mathbb{P}(s \leq s^-) = \left(\int_{s_0}^{s^+} ds \psi(x) \right) / \left(\int_{s^-}^{s^+} dx \psi(x) \right). \quad (33)$$

with $\psi(x) = \exp\left(\int_0^x -\frac{2\hat{\mu}}{\hat{\sigma}^2} ds\right)$. Eq. (33) has the following closed form solution:

$$\mathbb{P}(s \leq s^-) = \left(\exp\left[-\frac{2\hat{\mu}s_0}{\hat{\sigma}^2}\right] - \exp\left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2}\right] \right) / \left(\exp\left[-\frac{2\hat{\mu}s^-}{\hat{\sigma}^2}\right] - \exp\left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2}\right] \right), \quad (34)$$

with $\hat{\mu} = p\mu^+ + (1-p)\mu^-$ and $\hat{\sigma}^2 = p\left[(\mu^+ - \hat{\mu})^2 + \frac{\sigma^2}{m}\right] + (1-p)\left[(\mu^- - \hat{\mu})^2 + \frac{\sigma^2}{m}\right]$. \square

Proposition 2. The conditional systemic default probability, conditional to a given market trend has the following closed form solutions:

$$\begin{cases} q = \left(\exp\left[-\frac{(2\mu^-)s_0}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^-)s^+}{\sigma^2/m}\right] \right) / \left(\exp\left[-\frac{(2\mu^-)s^-}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^-)s^+}{\sigma^2/m}\right] \right), \\ g = \left(\exp\left[-\frac{(2\mu^+)s_0}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^+)s^+}{\sigma^2/m}\right] \right) / \left(\exp\left[-\frac{(2\mu^+)s^-}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^+)s^+}{\sigma^2/m}\right] \right). \end{cases}$$

Proof. From the proof of Proposition 1, we can derive immediately that during a downtrend, Eq. (33) yields the conditional default probability given a downtrend

$$q := \mathbb{P}(\text{default} \mid \mu^-) = \mathbb{P}(s \leq s^- \mid \mu^-)$$

with the following closed form solution:

$$q = \left(\exp\left[-\frac{(2\mu^-)s_0}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^-)s^+}{\sigma^2/m}\right] \right) / \left(\exp\left[-\frac{(2\mu^-)s^-}{\sigma^2/m}\right] - \exp\left[-\frac{(2\mu^-)s^+}{\sigma^2/m}\right] \right). \quad (35)$$

During an uptrend, Eq. (33) yields the conditional default probability given an uptrend:

$$g := \mathbb{P}(\text{default} \mid \mu^+) = \mathbb{P}(s \leq s^- \mid \mu^+)$$

with the following closed form solution:

$$g = \left(\exp \left[-\frac{(2\mu^+)s_0}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) / \left(\exp \left[-\frac{(2\mu^+)s^-}{\sigma^2/m} \right] - \exp \left[-\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right). \quad (36)$$

□

Proposition 3. Consider a debt-financed portfolio subject to a given default threshold as in the model described so far. Then, the default probability decreases with risk diversification in case of asset prices uptrend and increases with diversification in case of asset prices downtrend.

Proof. We provide an asymptotic analysis that explains the results presented in Proposition

3. Let rewrite Eq. (35) and Eq. (36) as

$$\begin{aligned} q &= \frac{M_{(-)}^{s_0} - M_{(-)}^{s^+}}{M_{(-)}^{s^-} - M_{(-)}^{s^+}} \\ &= M_{(-)}^{(s_0-s^+)} \left[M_{(-)}^{(s^- - s^+)} - 1 \right] - \frac{1}{M_{(-)}^{(s^- - s^+)} - 1} \end{aligned} \quad (37a)$$

$$\begin{aligned} g &= \frac{M_{(+)}^{s_0} - M_{(+)}^{s^+}}{M_{(+)}^{s^-} - M_{(+)}^{s^+}} \\ &= M_{(+)}^{(s_0-s^+)} \left[M_{(+)}^{(s^- - s^+)} - 1 \right] - \frac{1}{M_{(+)}^{(s^- - s^+)} - 1} \end{aligned} \quad (37b)$$

with $M_{(-)} = \exp \left[\frac{-(2\mu^-)m}{\sigma^2} \right]$ and $M_{(+)} = \exp \left[\frac{-(2\mu^+)m}{\sigma^2} \right]$. Then, the following result is straightforward:

$$\begin{cases} \lim_{m \rightarrow +\infty} q = 0 - (-1) = 1 \\ \lim_{m \rightarrow +\infty} g = 0 - 0 = 0 \end{cases}$$

To conclude, for any arbitrary small number $\epsilon \in (0, 1)$

$$\exists \bar{m} > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > \bar{m}$$

□

Proposition 4. Assume homogeneity in the banks' capital structure and the presence of a network of interbank exposures among banks. For a given default threshold, *ceteris paribus*, the leverage of an individual bank, measured in terms of its leverage ϕ , increases non-linearly with the level of its borrowers leverage and proportionally to the level of its interbank exposures.

Proof. To prove Proposition 4, we study the impact of the leverage of direct counterparties of bank i , ϕ_k on Eq. (10) :

$$\phi_i = l_i / \left(\sum_j z_{ij} \nu_j + \sum_k w_{ik} \left(\frac{\eta_k}{1 + r_f + \beta \phi_k} \right) \right). \quad (38)$$

Considering only the interbank assets:

$$\hat{\eta}_i = \sum_k w_{ik} \left(\frac{\eta_k}{1 + r_f + \beta \phi_k} \right). \quad (39)$$

We can prove the effect of the network on the leverage of banks simply by studying the sign and the relation between the derivative of $\hat{\eta}_i$ and ϕ_k :

$$\frac{\partial \hat{\eta}_i}{\partial \phi_k} = - \frac{w_{ik} \eta_k}{(1 + r_f + \beta \phi_k)^2} \quad (40)$$

As it comes out, the higher is ϕ_k the lower will be the value of interbank assets, $\hat{\eta}_i$. The derivative is proportional to w_{ik} , hence the greater the value of the exposures with the counterparties, the larger will be the effect of others' leverage on bank i interbank assets. We conclude simply noting that, as it is possible to see from Eq. (10), a decrease in the level of $\hat{\eta}_i$, induced by higher level of ϕ_k translates directly in higher level of ϕ_i for bank i , and hence of its leverage.

□

Proposition 5. Given the probability interval $\Omega_p := [0, \frac{1}{2})$ and under the existence of two possible states of the world where asset returns either follow a normally distributed downtrend

(or uptrend) with mean equal to μ^- (or μ^+) and variance σ , the banks' expected utility maximization problem, as formalized in Eq. (24), leads to the following alternative solutions:

1. If $s_0 < s^*$, the expected utility $\mathbb{E}U(\Pi_m)$ is maximized when $m^* = M$. Then, $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$, for all $m < M$.
2. if $s_0 \geq s^*$, there exists a subinterval $\Omega_{p^*} \subset \Omega_p$ s.t., to each $p^* \in \Omega_{p^*}$ corresponds an optimal level of diversification m^* in the open ball $B\left(\frac{1+M}{2}, r\right) = \left\{m^* \in \mathbb{R} \mid d\left(m^*, \frac{1+M}{2}\right) < r\right\}$ with center $\frac{1+M}{2}$ and radius $r \in [0, \alpha]$ where $\alpha = f(q, g)$. Then, $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$, for all $m \notin B\left(\frac{1+M}{2}, r\right)$.

Proof. Proposition 5.I

To prove Proposition 5.I, note first that the function to maximize, as formalized in Eq. (24), is not linear. Since what we are eventually interested in is the *max* of U , it is not possible to switch in general the linear operator with the expectation operator. To show the point, let us decompose the expected utility function for the two scenarios as follows:

$$U(\Pi_m)_{\mu^-} = q\pi^- + (1-q)\pi^+ - \frac{\lambda}{2}\{q[\pi^- - \mathbb{E}(\pi)]^2 + (1-q)[\pi^+ - \mathbb{E}(\pi)]^2\}; \quad (41a)$$

$$U(\Pi_m)_{\mu^+} = g\pi^+ + (1-g)\pi^- - \frac{\lambda}{2}\{g[\pi^- - \mathbb{E}(\pi)]^2 + (1-g)[\pi^+ - \mathbb{E}(\pi)]^2\}; \quad (41b)$$

Assuming for simplicity that $\lambda = 0$, the derivative Eq. 41a and Eq. 41b with respect to m , can be derived as follows:

$$\frac{\partial U(\Pi_m)_{\mu^-}}{\partial m} = \frac{\partial q}{\partial m}\pi^- - \frac{\partial q}{\partial m}\pi^+ = -\frac{\partial q}{\partial m}(\pi^+ - \pi^-); \quad (42)$$

$$\frac{\partial U(\Pi_m)_{\mu^+}}{\partial m} = \frac{\partial g}{\partial m}\pi^- - \frac{\partial g}{\partial m}\pi^+ = -\frac{\partial g}{\partial m}(\pi^+ - \pi^-); \quad (43)$$

Furthermore, given that utility function $U(\Pi_m)_\mu$ is the combination of the functions $U(\Pi_m)_{\mu^-}$ and $U(\Pi_m)_{\mu^+}$, it follows that its derivative is equal to the combination of the

derivatives of Eq. (42) and Eq. (43). Hence:

$$\begin{aligned}
\frac{\partial U(\Pi_m)_\mu}{\partial m} &= p \frac{\partial U(\Pi_m)_{\mu^-}}{\partial m} + (1-p) \frac{\partial U(\Pi_m)_{\mu^+}}{\partial m} \\
&= p \left[-\frac{\partial q}{\partial m} (\pi^+ - \pi^-) \right] + (1-p) \left[-\frac{\partial g}{\partial m} (\pi^+ - \pi^-) \right] \\
&= p \left(-\frac{\partial q}{\partial m} \Delta\pi \right) - (1-p) \left(\frac{\partial g}{\partial m} \Delta\pi \right)
\end{aligned}$$

where $\Delta\pi = \pi^+ - \pi^-$. In the above equations, the derivatives of the default probabilities q and g are respectively positive and negative. To prove it, we rewrite Eq. (35) and Eq. (36) as follows:

$$q = \frac{\exp \left[- (2m\mu^-)(s^+ - s_0) \right] - 1}{\exp \left[- (2m\mu^-)(s^+ - s^-) \right] - 1} \quad (44a)$$

$$g = \frac{\exp \left[- (2m\mu^+)(s^+ - s_0) \right] - 1}{\exp \left[- (2m\mu^+)(s^+ - s^-) \right] - 1} \quad (44b)$$

where, we discard the trivial case $\mu = 0$, and w.l.g. we set $\sigma^2 = 1$. The partial derivatives of (44a) and (44b) with respect to m are positive and negative, respectively:

$$\frac{\partial q}{\partial m} = \frac{2e^{-2m\mu^-(s_0+s^- - s^+)}\mu^- \left(e^{2m\mu^-s^+}(s^- - s_0) + e^{2m\mu^-s^-}(s_0 - s^+) + e^{2m\mu^-s_0}(-s^- + s^+) \right)}{e^{2m\mu^-(s^+ - s^-)}} > 0; \quad (45a)$$

$$\frac{\partial g}{\partial m} = \frac{2e^{-2m\mu^+(s_0+s^- - s^+)}\mu^+ \left(e^{2m\mu^+s^+}(s^- - s_0) + e^{2m\mu^+s^-}(s_0 - s^+) + e^{2m\mu^+s_0}(-s^- + s^+) \right)}{e^{2m\mu^+(s^+ - s^-)}} < 0. \quad (45b)$$

The signs of Eq. (45a) and Eq. (45b) follow from Jensen's inequality, which states that if U is convex function and X is any random variable, $U\{\mathbb{E}(X)\} \leq \mathbb{E}\{U(X)\}$. The inequality

is strict if U is strictly convex and X is non-degenerate. On the other hand, by assuming for simplicity that $s_0 = (s^+ - s^-)/2$, we can rewrite the second term in the product of the numerator in Eq. (45a):

$$e^{2m\mu^+s^+} \left(-\frac{\Delta s}{2}\right) + e^{2m\mu^+s^-} \left(-\frac{\Delta s}{2}\right) + e^{2m\mu^+s_0}(\Delta s) \quad (46)$$

where $\Delta s = s^+ - s_0 = s_0 - s^-$ by definition. Collecting then Δs :

$$\Delta s \left(\frac{1}{2}e^{2m\mu^+s^+} - \frac{1}{2}e^{2m\mu^+s^-} + e^{2m\mu^+s_0} \right) \quad (47)$$

Since the exponents of the terms in 47 are functions of s^+ , s^- and s_0 respectively, we can rewrite it in the following form:

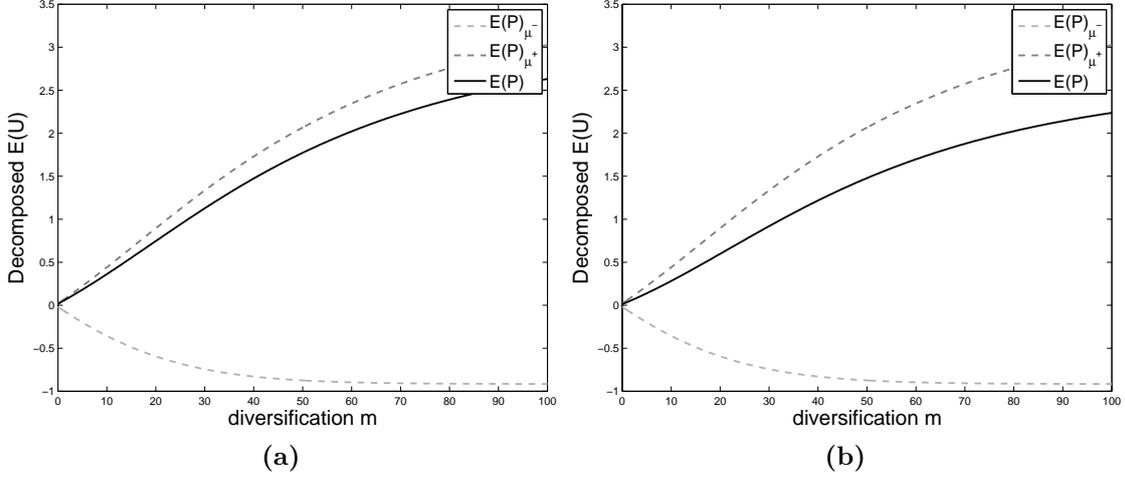
$$\Delta s \left(\frac{1}{2}e^{f(s^+)} - \frac{1}{2}e^{f(s^-)} + e^{f(s_0)} \right) \quad (48)$$

Notice that the above function is convex, because linear combination of convex functions. From Jensen's Inequality therefore:

$$\frac{f(s^+) + f(s^-)}{2} > f\left(\frac{s^+ + s^-}{2}\right) \quad (49)$$

The last result implies that the sign of the numerator of Eq. (45a) is positive, and hence the first derivative of q with respect to m will be also positive. Moreover, because of the symmetry between the two derivatives of g and q , it follows that the first derivative of g with respect to m is negative.

Figure 10: Conditional Expected Utility (Low Values of p).



Conditional expected utility. Parameter: **(a)** $p = 0.1$, **(b)** $p = 0.2$. Parameters: $\sigma^2 = 0.5$, $r_f = 0.001$, $\varepsilon = 0.1$, $\lambda = 0.1$, $\beta = 0.2$, $l = 0.5$, $m \in \{1, \dots, 100\}$, $s_0 = 1$, $w = 1$, $|\mu^+| = |\mu^-| = 0.003$.

The proof is complete once we show that the utility function U can be maximized at the extreme points of the feasible range of m . Given the non-linear dependence of $\partial g/\partial m$ and $\partial q/\partial m$, we prove *by existence* that the statement is valid. Since $p \in [0, 0.5)$, we can show that:

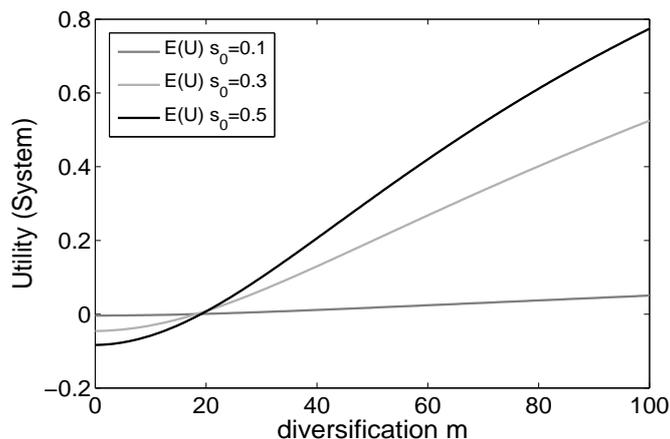
$$\begin{aligned}
 p = 0 : \frac{\partial U(\Pi_m)_\mu}{\partial m} &= -\frac{\partial g}{\partial m} \Delta\pi \geq 0 \\
 p \rightarrow 0.5 : \frac{\partial U(\Pi_m)_\mu}{\partial m} &= \frac{1}{2} \frac{\partial q}{\partial m} \Delta\pi - \frac{1}{2} \frac{\partial g}{\partial m} \Delta\pi = 0.
 \end{aligned}$$

As it turns out, considered the full interval of $p \in [0, 0.5)$ we found that $\left(\frac{\partial U(\Pi_m)_\mu}{\partial m}\right)_{p=0} > \left(\frac{\partial U(\Pi_m)_\mu}{\partial m}\right)_{p \rightarrow 0.5}$. Therefore, bank expected utility function will be always increasing monotonically with the level of diversification.

The last results imply that for values of p sufficiently small, the solution to the bank utility maximization problem, $\max_m U(\Pi_m)_\mu = U(\Pi_m)_{\mu^+}$, will be such that $m^* = M$ (see Figure 10). To conclude, by comparing Figure 7 and Figure 11, we show, by existence, that

extreme solutions can be obtained for values in the lower region of the parameter space of s_0 . One could therefore conjecture the existence of a generic threshold s^* below which the statement holds. See Proof *Proposition 5 II*. Given Eq. 10, these conditions will correspond to healthy banks with low levels of leverage and with a relatively large asset size.

Figure 11: Conditional Expected Utility (Equally Likely Scenarios)



Comparison between expected utility functions for different of s_0 . Parameters: $s_0 \in \{0.1, 0.3, 0.5\}$, $l = 0.5$, $p \rightarrow 0.5$, $\beta = 0.2$, $m \in \{1, 2, \dots, 100\}$, $\epsilon = 0.1$, $w = 1$, $r_f = 0.001$, $\lambda = 0.1$, $|\mu^+| = |\mu^-| = 0.003$.

□

Proof. Proposition 5.II

Point II in Proposition 5 shows that in a (arbitrage free) complete market of assets with a stochastic trend, even in the absence of transaction costs banks can maximize their MV utility by selecting an intermediate level of diversification m^* . To prove it, we will use the results of the previous proof and consider once again the two simplified equations of g and q in 44 and derive the partial derivatives with respect to s_0 . We can easily find that these are negative and positive respectively:

$$\frac{\partial q}{\partial s_0} = -\frac{2e^{2m\mu^-(s^+-s_0)}m\mu^-}{-1 + e^{2m\mu^-(s^+-s^-)}} < 0; \quad (50a)$$

$$\frac{\partial g}{\partial s_0} = -\frac{2e^{2m\mu^+(s^+-s_0)}m\mu^+}{-1 + e^{2m\mu^+(s^+-s^-)}} > 0. \quad (50b)$$

Now, we decompose the MV utility in Eq. (24) as follows:

$$\mathbb{E}U(\Pi_m)_{\mu^-} = p \left[(q\pi^- + (1-q)\pi^+) - \frac{\lambda}{2} \left(q(\Delta_-^-)^2 + (1-q)(\Delta_-^+)^2 \right) \right], \quad (51a)$$

$$\mathbb{E}U(\Pi_m)_{\mu^+} = (1-p) \left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2} \left(g(\Delta_+^-)^2 + (1-g)(\Delta_+^+)^2 \right) \right] \quad (51b)$$

$$\text{where } \begin{cases} \pi^+ := s^+ - s_0, \\ \pi^- := s^- - s_0, \end{cases}$$

$$\text{and } \begin{cases} \Delta_-^- := \pi^- - p[q\pi^- + (1-q)\pi^+], \\ \Delta_-^+ := \pi^+ - p[q\pi^- + (1-q)\pi^+], \\ \Delta_+^- := \pi^- - (1-p)[g\pi^- + (1-g)\pi^+], \\ \Delta_+^+ := \pi^+ - (1-p)[g\pi^- + (1-g)\pi^+]. \end{cases}$$

Let observe that Eq. (51a) is decreasing in m , while Eq. (51b) is increasing in m . Conversely, Eq. (51a) is increasing in s_0 and Eq. (51b) is decreasing in s_0 :

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^-}}{\partial m} = p \left(\frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0 \quad (52a)$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^+}}{\partial m} = (1-p) \left(\frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) > 0 \quad (52b)$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^-}}{\partial s_0} = p \left(\frac{\partial q}{\partial s_0} \right) (\pi^- - \pi^+) > 0 \quad (52c)$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^+}}{\partial s_0} = (1-p) \left(\frac{\partial g}{\partial s_0} \right) (\pi^- - \pi^+) < 0 \quad (52d)$$

Eq. (24) in Section 3.4 can be interpreted as a linear combination of Eq. (51a) and Eq. (51b) that are weighted by p and $(1 - p)$, respectively.

As it turns out, the optimum will be a function of both s_0 and m . The conditions to be verified in order to find the probability p^* that makes the partial derivatives equivalent are:

$$\text{FOC 1): } p \left(\frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) = (1 - p) \left(\frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) \quad (53)$$

$$\text{FOC 2): } p \left(\frac{\partial q}{\partial s_0} \right) (\pi^- - \pi^+) = (1 - p) \left(\frac{\partial g}{\partial s_0} \right) (\pi^- - \pi^+). \quad (54)$$

Putting into system the two equations, it must be true that:

$$\frac{\partial q}{\partial m} = \frac{\partial g}{\partial m} \quad \text{and} \quad \frac{\partial q}{\partial s_0} = \frac{\partial g}{\partial s_0} \quad (55)$$

The conditions are satisfied for all

$$p^* = 1 / \left(1 + \frac{\partial q}{\partial g} \right) \in \Omega_{p^*} \subset \Omega_P := [0, 1] \quad (56)$$

with $q = q(m^*)$, $g = g(m^*)$. In a more general form, Eq. (56) can be written as $p^* = f[g(m^*, s^*); q(m^*, s^*)]$. Since g , q and f are all one-to-one, for the Inversion Function Theorem are invertible functions. Hence, for a fixed value of p^* , must exists m^* and s^* such that simultaneously:

$$m^* = [(g^{-1}; q^{-1}) \circ f^{-1}] (p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{m^*}) \geq \mathbb{E}U(\Pi_m) \quad \forall m \geq m^*.$$

and:

$$s^* = [(g^{-1}; q^{-1}) \circ f^{-1}] (p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{s^*}) \geq \mathbb{E}U(\Pi_s) \quad \forall s \geq s^*.$$

The economic growth condition implies $p^* < \frac{1}{2}$. From (56), this is equivalent to write:

$$\begin{aligned} \frac{1}{2} &> \frac{\frac{\partial g}{\partial m}}{\frac{\partial q}{\partial m} + \frac{\partial g}{\partial m}} \\ \frac{\partial g}{\partial m} &< \frac{1}{2} \left[\frac{\partial q}{\partial m} + \frac{\partial g}{\partial m} \right] \\ \frac{\partial g}{\partial m} + \frac{\partial g}{\partial m} &< \frac{\partial q}{\partial m} + \frac{\partial g}{\partial m} \\ \frac{\partial g}{\partial m} &< \frac{\partial q}{\partial m} \end{aligned}$$

which is always true because from Eq. (45a)–(45b) $\frac{\partial g}{\partial m} < 0$ and $\frac{\partial q}{\partial m} > 0$. The last result implies that to each $p^* \in \Omega_{p^*} \subset \Omega_p := [0, \frac{1}{2})$ corresponds an optimal level of diversification m^* in the open ball $B(\frac{1+M}{2}, r) = \{m^* \in \mathbb{R} \mid d(m^*, \frac{1+M}{2}) < r\}$ with center $\frac{1+M}{2}$ and radius $r \in [0, \alpha]$ where $\alpha = f(q, g)$. Then, $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$, for all $m \notin B(\frac{1+M}{2}, r)$.

To conclude, by comparing similarly to before Figure 7 and Figure 11, we show by existence, that conditional to an initial price value above a *given* threshold s^* , the solution to the bank maximization problem is internal.

□

Proposition 7. If banks do not internalize the impact of social costs in the case of their default, the banking system will be over-diversified in external assets w.r.t. to the level of diversification that is socially desirable:

$$m^* \geq m^r \tag{57}$$

and

$$m^r < \infty \tag{58}$$

$$\forall s_0 \in (s^-, s^+).$$

Proof. Following the same line of reasoning used in the previous proof, we decompose

$\mathbb{E}U_r(\Pi_m)$ as follows:

$$\mathbb{E}U_r(\Pi_m)_{\mu^-} = p \left[(qk\pi^- + (1-q)\pi^+) - \frac{\lambda}{2} \left(q(k\Delta_-^-)^2 + (1-q)(\Delta_-^+)^2 \right) \right] \quad (59a)$$

$$\mathbb{E}U_r(\Pi_m)_{\mu^+} = (1-p) \left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2} \left(g(\Delta_+^-)^2 + (1-g)(\Delta_+^+)^2 \right) \right]. \quad (59b)$$

Eq. (59b) is not affected by Assumption 1 in Section 2.4 and remains equivalent to (51b).

Hence, the partial derivative with respect to m of Eq. (59b) is equal to Eq (52b):

$$\frac{\partial \mathbb{E}U_r(\Pi_m)_{\mu^+}}{\partial m} \equiv \frac{\partial \mathbb{E}U(\Pi_m)_{\mu^+}}{\partial m} = (1-p) \left(\frac{\partial g}{\partial m} \right) (\pi^- - \pi^+).$$

However, because of the factor k , the partial derivative with respect to m of Eq. (59a) is steeper than Eq (52a). It is easy to see that for any $k > 1$,

$$\frac{\partial \mathbb{E}U_r(\Pi_m)_{\mu^-}}{\partial m} = p \left(\frac{\partial q}{\partial m} \right) (k\pi^- - \pi^+) < \frac{\partial \mathbb{E}U(\Pi_m)_{\mu^-}}{\partial m} = p \left(\frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0$$

The condition to be verified is to find the probability p^r that makes the two equations to be equivalent:

$$\text{FOC: } (1-p) \left(\frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) = p \left(\frac{\partial q}{\partial m} \right) (k\pi^- - \pi^+).$$

Discarding the trivial solution $\mu_s = 0$, the condition is satisfied for all

$$p^r = 1 / \left(1 + \left(\frac{\partial q}{\partial g} \right) v \right) \in \Omega_{P^r} \subset \Omega_P := [0, 1] \quad (60)$$

with $q = q(m^r)$, $g = g(m^r)$ and $v = \frac{(k\pi^- - \pi^+)}{(\pi^- - \pi^+)}$.

In a more general form, (60) can be written as

$$p^r = f(v[g(m^*); q(m^*)]).$$

Since g , q , v and f are all one-to-one and hence invertible functions, for a fixed value of p^r ,

must exist an m^r such that

$$m^r = [(g^{-1}; q^{-1}) \circ v^{-1} \circ f^{-1}](p^r).$$

Eq.(60) is a decreasing function with respect to v which is a constant function bigger than one because $k > 1$. Then, $p^r < p^*$. This implies that the diversification level m^r is lower than the level m^* :

$$m^r = [(g^{-1}; q^{-1}) \circ v^{-1} \circ f^{-1}](p^r) < m^* = [(g^{-1}; q^{-1}) \circ f^{-1}](p^*)$$

From the proof of point II of Proposition 5 it follows directly that in case of uptrend and in presence of social costs, if for the individual bank is optimal to pursue maximal level of diversification, in general it is true that $m^r < \infty$, as the costs in the social optimum in case of systemic default will be of too great magnitude.

□

Proposition 6. Denote the probability of a systemic event conditional to the default of bank j by $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1)$. For any value of interbank exposure $1 \geq w \geq 0$ and some $j \in \{1, \dots, N\}$, it holds:

If $m = 1$: $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1) \rightarrow 0$ for $n \rightarrow \infty$;

If $m = M$: $\mathbb{P}(\phi \geq 1 | \phi_j \geq 1) = 1$ for $n \rightarrow \infty$.

Proof. In the case $m = 1$ and $1 \geq w \geq 0$, for $n \rightarrow \infty$ the portfolio of banks will be independent from each other, while the exposure to the interbank equals $\frac{w}{n}$, which tends to zero for large n . It follows that probability of several banks to default at the same time goes to zero because their shocks are independent and the exposure of a bank to any single counterparty tends to zero. Therefore, the systemic probability of default tends to zero.

In the case $1 \geq w \geq 0$ and $m = M$, banks will suffer exactly the same shock. Because of the assumption of homogeneity in the balance sheet structure of banks, if one bank defaults

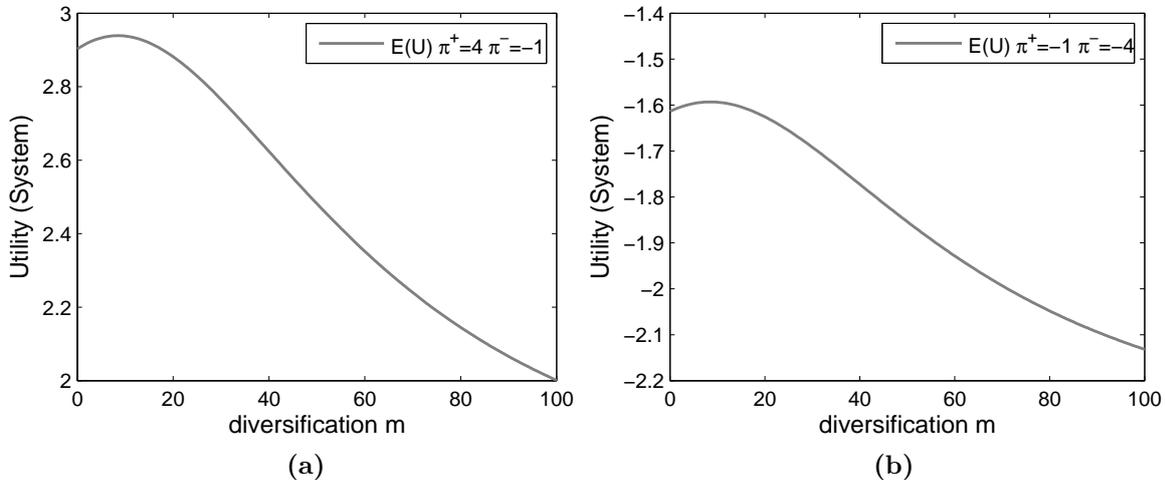
also all the other banks in the system must default by definition, regardless of the interbank exposure.

2.7 Validity of the default probability definition

The concept of default probability based on the first passage time models is widely used in credit risk models. In our framework we extend the standard default probability definition adopted in first passage time models where a firm goes in default when its value a touches a lower default boundary l . l , in general, represents the book values of the firm's debts. Instead of using the value of the firm, as driving variable, we use the leverage, defined as: $\phi = l/a$. This is an index bounded between 0 and 1. For this reason, we need to take into account not only the default boundary 1 but also the safe boundary 0 which naturally emerges because of the change of variable. The safe boundary represents a status of "permanent" strength as vice versa, the default boundary represents a status of "permanent" stress from which one firm cannot recover. Therefore, as we will argue here below, our change of variable does not deviate our notion of default probability from the standard one.

A bank could in principle reach at time $t^i > 0$ the safe boundary ε and then only later at time $t^{ii} > t^i > 0$ reach the default boundary at 1. Notice, however, that the probability of such an event is very low and this may occur only in a scholastic example when: 1) the variance σ is much larger than the trend μ ; 2) the two barriers $s^+ - s^-$ are very closed to each other; and 3) the initial value of external assets s_0 is very close to the boundaries.

Figure 12: Expected Utility fixing maximal losses and maximal profits.



Comparison of the utility function eliminating the dependence of the payoff structure from the upper and lower barrier. **(a)** $\pi^+=1$, $\pi^- = -1$; **(b)** $\pi^+ = 1$, $\pi^- = -4$. Parameters: $\sigma^2 = 0.5$, $l = 0.5$, $s_0 = 3.7$, $p = .4$, $\beta = 0.2$, $r_f = 0.001$, $\epsilon = 0.1$, $m \in \{1, \dots, 100\}$.

To test the conditions mentioned above, we performed a simple numerical test. In Figure 12, we plot the utility function of a bank by using fixed values of π^- and π^+ . In so doing, we eliminate the dependence of the expected payoffs, in the two scenarios considered in the model, from the distance of s_0 from the upper safe boundary. We test both for the case in which maximal losses π^- is larger than maximal profits π^+ and for the symmetric case.

The results confirm that the presence of an additional safe boundary in our framework does not have a significant impact on the optimal diversification strategy of banks in the mean-variance utility maximization problem.

2.8 Monte Carlo Simulations

In order to investigate how the conditional probability of systemic default varies with the diversification m , for a given density of the interbank network, we perform a set of Monte Carlo simulations.²⁹In a nutshell:

²⁹The simulation procedure is described in more detail in 2.8.1.

- We assume that, at every time step, each external asset is subject to an idiosyncratic shock. Because banks hold portfolios of such assets, their portfolio are subject to the corresponding combination of shocks.
- As diversification of banks increases, the overlap between portfolios also increases and so does the similarity of the shocks hitting banks' portfolios.
- When diversification is maximal, i.e., $m = M$, banks' portfolios are identical and are hit simultaneously by the same vector of shocks. In this extreme case, conditional to the default one bank, the probability of a systemic default is simply equal to one.

Unfortunately, because of correlation effects, there is no closed-form expression to describe mathematically the conditional systemic default probability (Frey and McNeil, 2003). Therefore, in our simulation framework we consider a system composed of N ($=25$) banks (with homogeneous balance sheet structure but heterogeneous portfolios of external assets) and approximate the conditional systemic default probability by the frequency of *simultaneous* defaults occurring after the default of the first bank. In practice, given the default of the first bank at time t_p , we test and measure the frequency of defaults among the $N - 1$ banks during two alternative time windows: 5 time steps t_{p+1}, \dots, t_{p+5} and 10 time steps $t_{p+1}, \dots, t_{p+5}, \dots, t_{p+10}$.

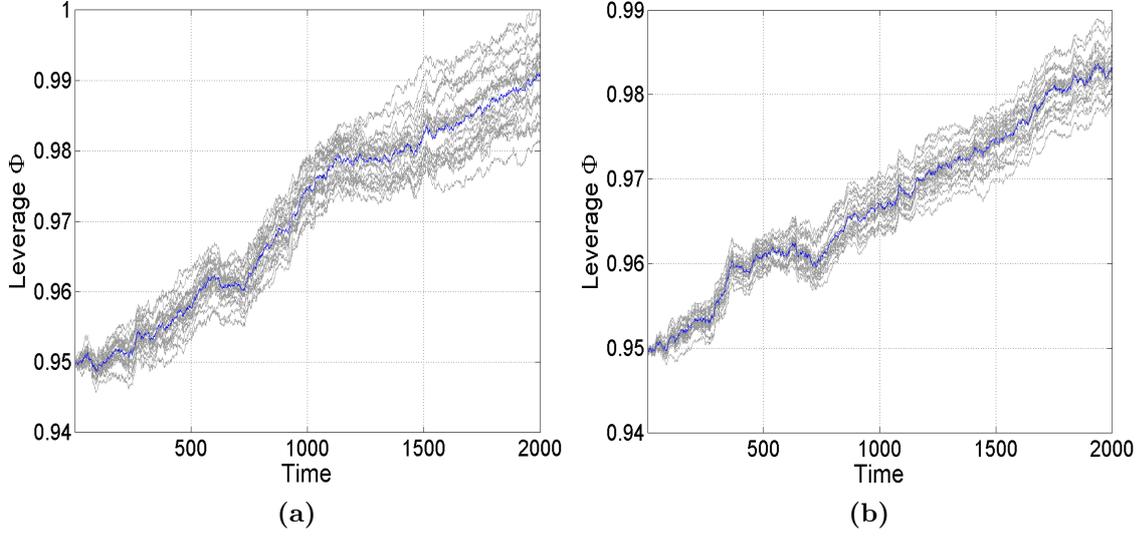
In order to perform our tests, we first simulate the Brownian motion in the scenario of a negative trend and compare the values of leverage, among the N banks, given a stochastic value of w and with other parameters (e.g., l , s_0) kept fixed. Under the assumption of homogeneity in the balance sheet structure of the banks, the leverage price-cycle used in the Monte Carlo analysis is specified as follows:

$$\begin{cases} \frac{ds_i}{s_i(t)} = \mu dt + \frac{\sigma}{\sqrt{m}} dB_p \\ \phi_i = \frac{l_i}{s_i + w_i \frac{l_i}{1+r_f + \beta \phi_j}} \end{cases} \quad (\text{C.1})$$

The trajectories of the leverages in two selected runs of the simulations are shown in Figure 13. Grey lines represent the leverage of each bank over time and the blue lines represent the average of the leverage values across $(N - 1)$ banks.

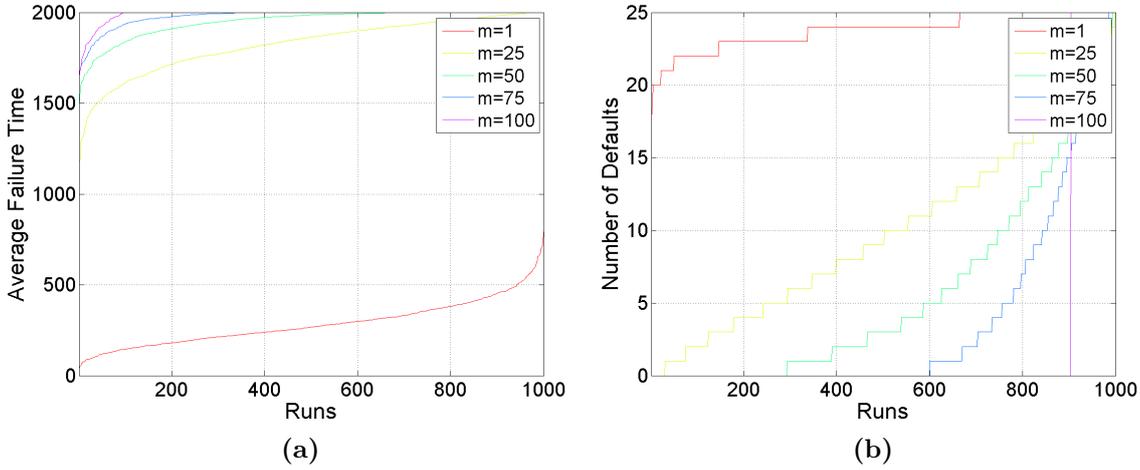
For a sample of 1000 runs, Figures 14 a) and b) show respectively the average time to default and the number of total defaults occurred in each run of the simulation at different diversification levels, $m \in \{1, 25, 50, 75, 100\}$. As expected, diversification brings immediate benefits even at low levels. Since we are interested in the conditional systemic default event, we count the number of simultaneous default. Namely, the number of banks that simultaneously collapse within a short time window, after an initial individual default. Specifically, the time window is set equal to 5 time steps in Figure 15 a) and to 10 time steps in Figure 15 b). For a number M ($=100$) of external assets, Figure 15 shows that the number of simultaneous defaults increases with the level of portfolio diversification. In particular, if all the banks were fully diversified, i.e., if $m=M$, all the N banks would default simultaneously.

Figure 13: Statistics about ϕ trajectories and banks' exit time.



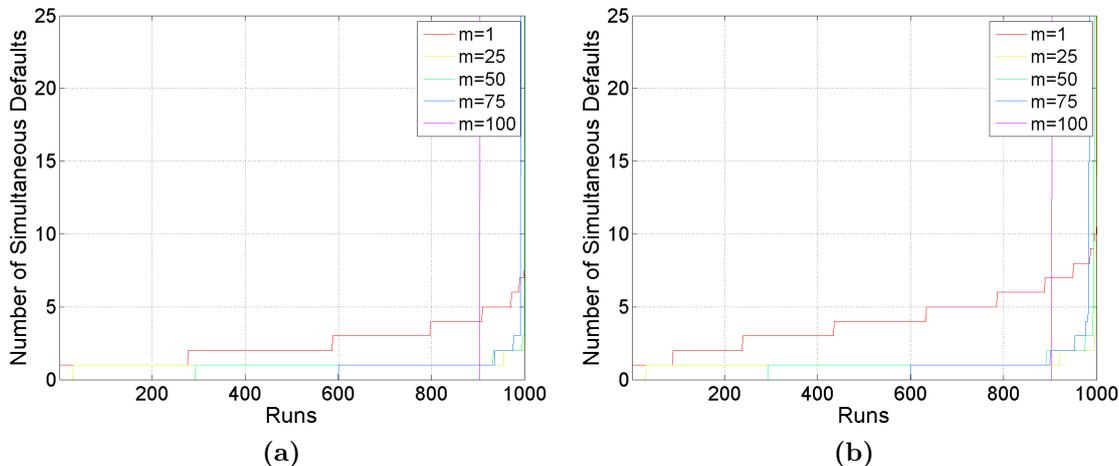
Trajectories and exit time obtained for n banks (grey lines) and their average (blue line) from the system of equations C.1. Parameters chosen for the simulation $\phi = 0.95$, number of banks $N = 25$, $\beta = 0.7$, $l = .5$, $\mu = -0.04$, $w = 0.5$. **(a)** Trajectories of banks ϕ for $m=50$; **(b)** Trajectories of banks ϕ for $m=75$.

Figure 14: Statistics on banks defaults.



Statistics on banks simultaneous defaults. Parameters are in both panels as in Figure 13. **(a)** Average failure time for $m \in \{1, 25, 50, 75, 100\}$; **(b)** Number of total defaults across banks for $m \in \{1, 25, 50, 75, 100\}$.

Figure 15: Statistics on banks defaults.



Statistics on simultaneous defaults. After one bank defaults, we keep track of the number of banks occurred within a certain time window. Parameters are set in both panels as in Figure 13. (a) Simultaneous defaults within 5 time steps; (b) Simultaneous defaults within 10 time steps.

2.8.1 Simulation Procedure

To verify numerically the Proposition 6, we study the dynamics of ϕ and the number of simultaneous defaults among the $N-1$ banks after the first bank goes in default.

Consider in particular to run many times a stochastic process describing the leverage of the banks in the system with an upper barrier at one and to restart it from the same initial condition $\phi_0 < 1$. We stop the process once all the banks exit this barrier.

Furthermore, we consider a discretized version of Brownian motion, we set the step-size as Δt and let B_p denote $B(t_p)$ with $t_p = p\Delta t$. According to the properties of Brownian motion, we find:

$$B_p = B_{p-1} + dB_j \quad p = 1, \dots, P.$$

where P denotes the number of steps that we take with t_0, t_1, \dots, t_P as a discretization of the interval $[0, T]$, and dB_p is a vector of normally distributed random variables $k \times 1$ with zero mean and variance Δt . In our case k corresponds to the number of external assets

held by the individual bank. Computing numerically Eq. (15) we can simulate a Brownian motion on the external assets prices.

To simulate the stochastic dynamical system of equations, we follow the simple Euler discretization scheme. We first discretize the dynamics of the processes $s_{i,t \geq 0}$. Then, we substitute the value of s and l into the discrete time version of the last expression into the above system of equations. It is unnecessary to derive the dynamics of $\phi_{i,t \geq 0}$ via Ito's Lemma from the dynamics of $s_{i,t \geq 0}$. The resulting discrete version of the system of equations above is:

$$\begin{cases} s_{i,p} &= s_{i,p-1} + \mu_s(\phi_{i,p-1}, s_{p-1}, \phi) \Delta t + \sigma(s_{i,p-1}) dB_p \\ \phi_{i,p} &= \frac{l_{i,p}}{s_{i,p} + w_{i,p} \frac{l_{i,p}}{1+r_f + \beta \phi_{j,p}}} \end{cases}$$

where $\phi_{i,p}$ and $s_{i,p}$ are the approximations to $\phi_i(p\Delta t)$ and $s_i(p\Delta t)$, while Δt is the step size, $dB_p = B_p - B_{p-1}$ and P is the number of steps we take in the process.

We run Monte Carlo simulations for a total of N sample paths within $[0, T]$, i.e. economy terminal date. From $t = 1$, we measure the time step T at which the process $\{\phi_i\}_{t \geq 0}$ crossed the default boundary fixed at one for each bank i . We stop the process when all the banks in the system reached this point and keep track of the fraction of trajectories that crossed the default boundary after the default of one bank in a given number of time steps. We make sure that first bank defaults at time T are not accounted for.

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3 Chapter: A model of network formation for the overnight interbank market

with Mikhail Anufriev(University of Technology of Sydney), Valentyn Panchenko (University of New South Wales) and Paolo Pin (Bocconi University)

Abstract

We introduce an endogenous network model of the interbank overnight lending market. Banks are motivated to meet the minimum reserve requirements set by the Central Bank, but their reserves are subject to random shocks. To adjust their expected end-of-the-day reserves, banks enter the interbank market, where borrowers decrease their expected cost of borrowing with the Central Bank, and lenders decrease their deposits with the Central bank in attempt to gain a higher interest rate from the interbank market, but face a counter-party default risk. In this setting, we show that a financial network arises endogenously, exhibiting a unique giant component which is at the same time connected but bipartite in lenders and borrowers. The model reproduces features of trading decisions observed empirically in the Italian e-MID market for overnight interbank deposits.

3.1 Introduction

This paper presents an endogenous network formation model of the overnight interbank lending market, where the equilibrium is able to consider endogenous choices of lending volumes and interest rates . Verifying our theoretical results with empirical evidence from the Italian electronic market for interbank deposits (e-MID), we find that our model can

successfully predict various network characteristics on a *daily* frequency. This provides a coherent framework to analyze frictions and market freezes in the interbank lending market, which since the Global Financial Crisis has emerged in the limelight of financial stability regulation.

Indeed, there has been a blooming literature highlighting the significance of interconnected financial institutions in the propagation of shocks throughout the financial system.³⁰ On the theoretical side, most papers have focused on the risk of contagion given different network structures. In Elliott, Golub, and Jackson (2014) the financial organizations are linked via cross-ownership of equity shares and the chain of defaults is triggered by discontinuous loss of firm's value falling below a threshold; in Cabrales, Gottardi, and Vega-Redondo (2014) different organizations are linked by investing to the same projects (i.e., by exchanging their assets) and the default of a project leads to the loss of value of several firms; in Glasserman and Young (2015) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) the financial institutions are linked through the interbank lending, so that a shortfall in assets of one bank can generate a cascade of incomplete repayments. In these papers, not only do the definition of interconnectedness and propagation mechanisms differ, the relative stability of different network architectures is also debated. Although seminal work by Allen and Gale (2000) indicate that most systemically stable networks are fully connected, recent literature has highlighted how dense connections can facilitate the transmission of large idiosyncratic shocks into systemic events.

However, any observed network is comprised of endogenous choices made by profit-maximizing financial entities. As Haldane and May (2011) point out, the most important channels of shock propagation may not be those that involve balance sheet connections (as in the financial contagion models cited above) but those that affect the process of creation

³⁰The network is *connected* if any node can be reached from any other node by following the links, in particular, directed links. If it is not connected, one can split the network on disjoint connected components and characterise the degree of connectivity on the basis of this split. On the other hand, within each connected component, one can look at the proportion of existing links with respect to all possible links. This proportion characterises *density* of the component. Component is called *fully connected*, if it has the largest possible density, i.e., all possible links are present.

or destruction of such connections. With such indirect channels, the shocks will be amplified via drops in market liquidity either due to a general price fall for some classes of assets or because of a higher expectations of counter-party defaults (Gai and Kapadia, 2010a). Furthermore, shocks may cause a diminishing availability of the interbank loans, as the banks are hoarding on liquidity (Gai and Kapadia, 2010b).³¹ A joint understanding of how the actual networks are formed as well as the implications of the resulting structure is therefore essential for correctly predicting the consequences of shocks.

This paper first rationalizes the formation of the Italian overnight interbank network and then derives implications from the endogenously formed network structure. Specifically, we adapt and extend the standard setup introduced in Poole (1968) to a decentralized bilateral exchange setting with counterparty risk. In the model, banks aim to maintain minimum reserve requirements. In the beginning of the period they can trade in a competitive interbank market to adjust their position, but face uncertainty due to late settlements. Outside options are provided by the Central Bank, which operates a corridor system for banks to borrow and lend from the Central Bank.³²

Whereas most theoretical work in network formation has been limited to simple stylized networks with categorical choice variables, reality is a continuum and the extent of potential instability and cascades is affected by not only the existence of links but also the amount and price of loans exchanged through each link. With this in mind, our model allows banks to choose link formation optimizing jointly on the direction, amount and interest rate of loans. The choice over price and volume dimensions is also accommodated in our novel equilibrium definition, which extends Jackson and Wolinsky (1996)'s seminal notion of pairwise stability. This allows us to generate an equilibrium bipartite network in which borrowers and lenders form a unique component without cycles in lending and borrowing. We also yield a rich set

³¹Endogenous formation of networks of financial institutions have also been studied in the working paper version of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), in the paper by Hommes, van der Leij, and in 't Veld (2014), Farboodi (2014) and by Babus and Hu (2015).

³²Other recent contributions in a similar setup include Afonso and Lagos (2015) and Bech and Monnet (2013) who model decentralized exchange as we do, but use the search and matching framework and do not consider the network setting.

of implications. For example, an increase in the Central Bank borrowing or lending rates or minimum reserve requirements leads to an increase in the average interbank market rate. We also show that large increases in default probability may lead to “market freeze” where no trade is feasible.

Compared to previous network formation variables limited to stylized settings, continuous choice variables allows our model predictions to better match empirical data. Indeed, when comparing our model predictions to the Italian overnight interbank market, e-MID, we find many consistencies. This contributes to the empirical literature on interbank lending, which has so far been largely descriptive in nature, by providing a sound theoretical framework for analyzing the empirically observed network characteristics. Previous work in this area includes Craig and Von Peter (2014), who document characteristics of the German interbank lending network and show that it follows a core periphery structure. van Lelyveld and in ’t Veld (2014) use the large exposure data for Dutch banks and also confirm a core-periphery setting.

The rest of this paper is organised as follows. In Section 3.2 we introduce the theoretical model of interbank lending decisions. The resulting endogenous decisions will induce a network of borrowing and lending. Sections 3.3 and 3.4 adapt the established equilibrium concept of pairwise stability to our context. This allows us to limit our attention to certain networks in equilibrium and to characterize them in terms of financial quantities and of network topology. Section 3.5 further refines our notion of pairwise stable equilibrium. Section 3.6 discusses the implications of monetary policy and the effects market conditions for our model. Section 3.7 provide some examples. Section 3.8 compares the predictions of our theoretical model with the empirical eMID data. Section 3.9 concludes.

3.2 Model

Consider a finite set \mathcal{N} of banks and the Central Bank. The Central Bank establishes the minimum reserve requirements on a fraction of banks deposits at the end of the day. We are not modeling deposits explicitly and simply denote the resulting requirement on the *reserve*

holdings of bank $i \in \mathcal{N}$ by T_i . If reserve holdings at the end of the day are lower than T_i , bank i must borrow the difference at a *penalty rate* r^p . If, instead, reserve holdings are in excess of the required level, bank deposits the excess reserves to the Central Bank at a *deposit rate* r^d . The interest rates r^p and r^d with $r^p > r^d$ efficiently establish the upper and lower bounds for the interbank interest rates and are called ceiling and floor in the corridor system.

We model one day of banks operations as follows. In the beginning of the day, every bank $i \in \mathcal{N}$ predicts the net cash position from operations by the end of the day, s_i . In other words, s_i is the projected reserve holdings at the end of the day in the absence of borrowing or lending from the overnight market. The banks cannot perfectly predict their reserve holdings, however. It means that the *actual* reserve holdings (net of adjustments from the interbank market) are $s_i + \varepsilon_i$, where ε_i is a realisation of the random variable whose distribution is known to the bank. During the day, the banks have an access to the decentralized interbank market for overnight funds, where they are able to lend or borrow from each other in order to optimize their expected profit by adjusting their projected position of reserves s_i . Let C_i denote this adjusted position, i.e., the expected level of reserve holdings after the lending/borrowing in the interbank market. Finally, at the end of the day, after the interbank market is closed, the uncertainty about the actual level of reserve holdings is realized and the bank either deposit excess funds to the Central Bank or borrows from it to cover the minimum reserve requirements.

Our main focus of analysis will be on the interbank market for overnight funds. In this market, every pair of banks may enter into the bilateral agreement, when, say, bank i agrees to lend to bank j a quantity $\ell_{ij} > 0$ with interest rate r_{ij} . We do not rule out *a priori* that i and j may have two agreements, when i lends to j and, at the same time, borrows from j .³³ We introduce the *lending matrix* $\mathbf{L} = \{\ell_{ij}\}_{i,j=1}^N$ with $\ell_{ij} \in \mathbb{R}_+$. (We set $\ell_{ii} = 0$ for all i .) This lending matrix induces the weighted directed network. There is a counter-party risk in the overnight market, i.e., with a small probability a borrower may default on its loan. To

³³We prove later that this is not possible in equilibrium.

focus exclusively on the overnight market we do not model the origins of defaults, assuming that these origins are outside of the interbank market. Specifically, we make the following assumption.

ASSUMPTION 1. *Bank i has a probability of default, $q_i \in [0, 1]$, which is independent of banks' reserves and is known to every bank in the market. Defaults of different banks are independent events.*

Given the borrowing-lending network, every bank's projected reserves towards the end of the day is

$$C_i = s_i - \underbrace{\sum_{k \in \mathcal{N}} \ell_{ik}}_{\text{Lending}} + \underbrace{\sum_{m \in \mathcal{N}} \ell_{mi}}_{\text{Borrowing}}. \quad (1)$$

The first sum is over all borrowers of bank i and the second sum is over all lenders to bank i . After the interbank market is closed, the uncertainty about the actual banks reserve holdings is resolved. The reserve holdings of bank i are given by

$$(s_i + \varepsilon_i) - \sum_{k \in \mathcal{N}} \ell_{ik} + \sum_{m \in \mathcal{N}} \ell_{mi} = C_i + \varepsilon_i$$

where ε_i is a realization of a continuous random variable with CDF function F_i and density f_i . Thus, $s_i + \varepsilon_i$ is the actual net cash position from usual banking operations, and ε_i can be interpreted as an error made in predicting this variable in the beginning of the day. We assume that the distribution F_i is known and its mean is 0. The expected payoff on the overnight position of bank i in the Central Bank is

$$\pi_i^{\text{CB}} = - \int_{-\infty}^{T_i - C_i} (T_i - (C_i + \varepsilon_i)) r^p dF_i(\varepsilon_i) + \int_{T_i - C_i}^{\infty} ((C_i + \varepsilon_i) - T_i) r^d dF_i(\varepsilon_i). \quad (2)$$

The first term gives an expected loss that the bank incurs by not meeting the reserve requirements and thus the integration is over the cases when $T_i > C_i + \varepsilon_i$. The second term integrates over the remaining instances and represents an expected gain of the bank for having excess reserve holdings.

Banks enter the interbank market in order to adjust their reserve holdings and thus affect their expected profits π^{CB} given by (2). In this market, borrowing banks bear interest expenses, whereas lending banks earn interest, but are subject to the counter-party default risk. In accordance with Assumption 1 the bank's default probability is determined by its overall standing and is not affected by simple overnight market operations. The next Assumption discusses the consequences of default for the interbank market.

ASSUMPTION 2. *If a borrower defaults, then the lender will not receive its loan (neither principal nor interest) back. If a lender defaults, then the borrower will have to pay back both principal and interest of a loan to some centralized institution.*

Under Assumption 2 the expected profit from the transactions on the interbank market of bank i is

$$\pi_i^{\text{IM}} = \underbrace{\sum_{k \in \mathcal{N}} r_{ik} \ell_{ik} (1 - q_k)}_{\text{interest due from repaying borrowers}} - \underbrace{\sum_{k \in \mathcal{N}} \ell_{ik} q_k}_{\text{losses because of defaulted borrowers}} - \underbrace{\sum_{m \in \mathcal{N}} \ell_{mi} r_{mi}}_{\text{interest due to lenders}} \quad (3)$$

Note an asymmetry in the profit which is a consequence of Assumption 2.³⁴ The first two terms represent the expected payoff from the bank's borrowers: if a borrower honors the loan, bank i gains in the interest paid on the loan, and if a borrower defaults, bank i loses its loan. The third term is the interest payment on own loans which bank i should make.

The total expected profit that bank i derives from the overnight market is then

$$\pi_i = \pi_i^{\text{CB}} + \pi_i^{\text{IM}}, \quad (4)$$

where the two terms are given by (2) and (3). Bank i receives this profit only when it does not default (i.e., with probability $1 - q_i$). As default is an exogenous event to the overnight

³⁴The assumption that if a borrower defaults, then the lender will not receive its cash (neither principal nor interest) back is made for simplicity. It can be easily relaxed by assuming that a defaulted borrower repays a fraction $\gamma \in [0, 1]$ of its obligations (so-called "hair-cut"). Then, expression (3) will still hold with q_k substituted by $(1 - \gamma)q_k$. An analogous version of the same assumption is also in the recent paper by Blasques, Bräuning, and Van Lelyveld (2015).

market operations, without loss of generality³⁵ we assume that the objective of bank i is to maximize π_i in (4).

As it is mentioned above, banks have a complete knowledge about probabilities of defaults of counter-party in the interbank market but are subject to uncertainty regarding final reserve holdings. In this environment, each of the banks is willing to maximise (4). This leads to the following trade-offs for borrowers and lenders. When bank i borrows some funds in the interbank market, it increases its own probability to meet the reserve requirements (and overall increases π^{CB}) but has to pay an interest on the loans (thus decreasing π^{IM}). Note that for the borrower the counter-party does not matter in this trade-off, as it should pay an interest anyway according to Assumption 2. For lenders the trade-off is more complicated. When bank i lends funds, it decreases its own probability to meet the reserve requirements (and the whole expression in (2)) but may increase or decrease the net cash flow from the borrowers (the first two terms in (3)) depending on the interest rate and the probability of counter-party default. This also means that for a lender the choice of counter-party matters.

3.3 Pairwise Stable Equilibrium Networks

Given exogenous variables, i.e., penalty rate r^p , deposit rate r^d , predicted net cash positions of different banks s_i , reserve requirements T_i , the CDF function F_i for the funds' uncertainty shock ε_i , and the banks probabilities of default q_i , the banks enter the interbank lending market and form bilateral agreements. The outcome is summarized by the lending matrix $\mathbf{L} = \{\ell_{ij}\}$ and by the matrix of the corresponding interest rates $\mathbf{r} = \{r_{ij}\}$ assigned to the positive elements of matrix \mathbf{L} . This structure induces the directed weighted network that we denote $g = (\mathbf{L}, \mathbf{r})$. Whenever $\ell_{ij} > 0$, we say that there is a directed link from i to j and that this link has two weights, positive amount ℓ_{ij} that bank i lends to bank j and the interest rate $r_{ij} \geq 0$ corresponding to this loan.

For a given network g we can calculate, using (1), the projected reserves of every bank

³⁵Let B_i be an exogenous payoff in the case if bank i defaults. Assumption 1 guarantees that maximization of $(1 - q_i)\pi_i + q_i B_i$ is equivalent to maximization of π_i .

towards the end of the day but before the uncertainty is resolved. We denote these reserves $C_i(g)$ and call them the *interim* reserves of bank i . Using (2), (3), and (4), we can compute the expected payoff of bank i , which is denoted as $\pi_i(g)$.

We will limit our attention to the networks that are consistent with banks' incentives to maximise their payoffs (4) taking into account existing network structure. In this sense, we consider *strategically stable* networks. Inspired by a milestone notion introduced by Jackson and Wolinsky (1996), we consider networks that are stable in the sense of the following definition.

DEFINITION 1. *Network $g = (\mathbf{L}, \mathbf{r})$ is a **pairwise stable equilibrium** if the following conditions are met:*

1. *no single bank i can remove an existing link in which it is involved (i.e., for any $k \in \mathcal{N}$ with $\ell_{ik} > 0$ and for any $m \in \mathcal{N}$ with $\ell_{mi} > 0$) and be better off;*
2. *no couple of banks i and j with link $\ell_{ij} > 0$, can change their agreement to $(\tilde{\ell}_{ij}, \tilde{r}_{ij})$ (keeping the rest of the network the same) that would make bank i better off without making bank j worse off;*
3. *no couple of banks i and j with $\ell_{ij} = 0$, can find an agreement $(\tilde{\ell}_{ij}, \tilde{r}_{ij})$ with $\tilde{\ell}_{ij} > 0$ that would make both banks i and j better off.*

The first requirement says that in the pairwise stable equilibrium network, any two linked banks find their deal profitable, that is none of them would prefer to remove the link unilaterally. The second requirement further specifies what type of the deal the banks are agree upon. Namely, it says that any bilateral agreement, in the equilibrium, is Pareto efficient for the two involved banks given the rest of the network. The third requirement says that banks have no incentive to add a new link to the network.

Comparing with a standard notion of pairwise stability as introduced in Jackson and Wolinsky (1996), our concept is an adaptation of their notion to a setting where every link have *three* payoff-relevant variables: the direction of the loan, the loan amount and

the interest rate. We assume that if a potential link between i and j may be created, the network configuration is not in equilibrium. Then, the process of link creation is interpreted as a bargaining process between banks i and j . The second requirement of Definition 1 provides some discipline on the outcome of this bargaining without making more specific assumptions about banks' behavior. Finally, as Jackson and Wolinsky (1996), we assume that every bank can unilaterally cut any existing link.

In the rest of this section we will characterize topological properties of network $g = (\mathbf{L}, \mathbf{r})$ that are consistent with pairwise stable equilibrium.

3.3.1 Payoff improving links and Feasibility sets

Let us begin by studying the incentives that the two individual banks have to create or sustain a link. For this purpose we introduce the following notation. For a given network g and for a pair of banks i and j with $\ell_{ij} > 0$, we denote as ij the link from i to j , and as $g - ij$ the network obtained from g by deleting this link. In the opposite situation when $\ell_{ij} = 0$ in network g , we denote $g + ij$ a network obtained from g by adding a link along which i lends to j some fixed and positive amount ℓ with some interest rate r . For the latter case, when a link between i and j is added, we introduce the quantities

$$\Delta_i^{g+ij}(\ell, r) = \pi_i(g + ij) - \pi_i(g),$$

and

$$\Delta_j^{g+ij}(\ell, r) = \pi_j(g + ij) - \pi_j(g),$$

that represent the changes in the expected profit of the lender and borrower, respectively, in the case if they would agree on link (ℓ, r) . Furthermore, let us introduce the *feasibility regions*, $F_{i \rightarrow j}^L(g)$ for bank i as for a lender to j , and $F_{j \leftarrow i}^B(g)$ for bank j as for a borrower from i , as sets

$$F_{i \rightarrow j}^L(g) = \{(\ell, r) : \ell > 0, r \geq 0, \Delta_i^{g+ij}(\ell, r) > 0\},$$

and

$$F_{j \leftarrow i}^B(g) = \{(\ell, r) : \ell > 0, r \geq 0, \Delta_j^{g+ij}(\ell, r) > 0\} .$$

These sets contain all pairs of ℓ and r which make lender i strictly better off from a link with j and borrower j strictly better off from a link with i , respectively.

Note that for the network g' , which is the same as g except that link (ℓ_{ij}, r_{ij}) is present, we can write the changes in the expected profit of the lender and borrower when this link is removed as

$$\pi_i(g') - \pi_i(g' - ij) = \Delta_i^{g+ij}(\ell_{ij}, r_{ij}), \quad \text{and} \quad \pi_j(g') - \pi_j(g' - ij) = \Delta_j^{g+ij}(\ell_{ij}, r_{ij}).$$

The notation is consistent with the previous one because the rest of the network is held the same.

With this notation, conditions 1 and 3 of Definition 1 can be reformulated as follows. The first condition requires for a pairwise stable network g' , where link ij (with $\ell_{ij} > 0$ and $r_{ij} \geq 0$) is present, that for $g = g' - ij$ it is

$$\Delta_i^{g+ij}(\ell_{ij}, r_{ij}) \geq 0 \quad \text{and} \quad \Delta_j^{g+ij}(\ell_{ij}, r_{ij}) \geq 0 .$$

Jointly these two inequalities mean that the pair (ℓ_{ij}, r_{ij}) lies in the intersection of the closures of feasibility regions, $\text{Cl } F_{i \rightarrow j}^L(g) \cap \text{Cl } F_{j \leftarrow i}^B(g)$, for the network $g = g' - ij$. The third condition requires for a pairwise stable network g with non-existing link $\ell_{ij} = 0$ to have no pair of (ℓ, r) such that

$$\Delta_i^{g+ij}(\ell, r) > 0 \quad \text{and} \quad \Delta_j^{g+ij}(\ell, r) > 0 .$$

This means that the intersection of feasibility regions must be empty $F_{i \rightarrow j}^L(g) \cap F_{j \leftarrow i}^B(g) = \emptyset$.

Our next natural step is to analyze how the feasibility sets of lender i and borrower j look like for an arbitrary pair of banks in a given network, conditional on the rest of the network. The following quantity will play an important role in the analysis. For given network g and

every bank i define

$$W_i = r^d + (r^p - r^d) F_i(T_i - C_i) = r^p F_i(T_i - C_i) + r^d(1 - F_i(T_i - C_i)). \quad (5)$$

This is the *expected marginal rate* of bank i 's transaction with the Central Bank. When bank i decides to lend an extra reserves to any other bank in the interbank market, the profit of i from the Central Bank interactions will expect to drop marginally by W_i . Analogously, when bank i decides to borrow extra reserves from any other bank in the interbank market, the profit of i from the Central Bank interactions will expect to increase marginally by W_i . The following lemma summarizes some simple but useful properties of the expected marginal rate of a bank in the network.

LEMMA 1. *Consider an arbitrary (not necessarily equilibrium) network g and bank i with reserve requirements T_i . Then:*

1. $r^d \leq W_i \leq r^p$.
2. W_i is a decreasing function of the interim reserves C_i . In particular, it increases with extra lending that i makes and decreases with extra borrowing it makes.
3. $W_i \rightarrow r^p$ when $C_i \rightarrow -\infty$ and $W_i \rightarrow r^d$ when $C_i \rightarrow +\infty$.

Now let us take two banks i and j from network g and fix the rest of the network. We analyze all possible loans (ℓ, r) that i may offer to j , i.e., any $\ell \geq 0$. Note that both W_i and W_j depend on ℓ but do not depend on r . We will write this dependence explicitly as $W_i(\ell)$ and $W_j(\ell)$. Lemma 1 implies that $W_i'(\ell) > 0$ and $W_j'(\ell) < 0$.

We characterize the feasibility sets for a loan between lender i and borrower j as follows.

LEMMA 2. *Consider an arbitrary network g and banks i and j . Let bank i consider to lend reserves to bank j , and let the rest of the network be fixed. Then:*

1. *feasibility set of the lender is the strict epigraph of function $h^L : \mathbb{R}^+ \rightarrow \mathbb{R}$, i.e.,*

$$F_{i \rightarrow j}^L(g) = \{(\ell, r) : \ell > 0, r > h^L(\ell)\}.$$

Moreover, function h^L is strictly increasing and

$$h^L(0) = \frac{q_j + W_i(0)}{1 - q_j}. \quad (6)$$

2. feasibility set of the borrower is the strict hypograph of function $h^B : \mathbb{R}^+ \rightarrow \mathbb{R}$, i.e.,

$$F_{j \leftarrow i}^B(g) = \{(\ell, r) : \ell > 0, r < h^B(\ell)\}.$$

Moreover, function h^B is strictly decreasing and

$$h^B(0) = W_j(0). \quad (7)$$

Proof. We use the result of Lemma 1 and the derivatives of the payoff functions of lender and borrower as computed in Appendix 3.10. Note that the derivatives of function $\Delta_i^{g+ij}(\ell, r)$ coincide with the derivatives of $\pi_i(\ell, r)$ when i lends to j funds ℓ with interest rate r , and so $\partial \Delta_i^{g+ij} / \partial \ell = r(1 - q_j) - q_j - W_i(\ell)$ and $\partial \Delta_i^{g+ij} / \partial r = 1 - q_j$. Similarly, the derivatives of function $\Delta_j^{g+ij}(\ell, r)$ coincide with the derivatives of $\pi_j(\ell, r)$ when j is a borrower of funds ℓ with interest rate r from i , and so $\partial \Delta_j^{g+ij} / \partial \ell = -r + W_j(\ell)$ and $\partial \Delta_j^{g+ij} / \partial r = -\ell$.

Consider lender i and start with the boundary of the feasibility set where $\ell = 0$. On this boundary $\Delta_i^{g+ij}(\ell, r) \equiv 0$ and $\partial \Delta_i^{g+ij} / \partial \ell = r(1 - q_j) - q_j - W_j(0)$. This derivative is positive for $r > r^*$, where threshold r^* is defined by the right hand-side of (6); and it is negative for $r < r^*$. Consider the case, when $r < r^*$. Then for any $\ell > 0$ we have $\partial \Delta_i^{g+ij} / \partial \ell = r(1 - q_j) - q_j - W_j(\ell) < r(1 - q_j) - q_j - W_j(0) < 0$. In other words, function Δ_i^{g+ij} starts with value 0 and strictly decreases with ℓ in this case. Therefore, all points (ℓ, r) with $r < r^*$ are outside of the feasibility set.

Let us now fix arbitrary $\tilde{\ell} > 0$. We have established that when r is sufficiently small, $\Delta_i^{g+ij}(\tilde{\ell}, r) < 0$. But Δ_i^{g+ij} is a strictly increasing function of r , converging to $+\infty$ when $r \rightarrow \infty$, according to (3) (and because (2) does not depend on r). Therefore, there exists a unique value of $\tilde{r} > 0$ for which $\Delta_i^{g+ij}(\tilde{\ell}, \tilde{r}) = 0$. Point $(\tilde{\ell}, \tilde{r})$ thus belongs to the boundary of

the feasibility set, whereas all the points $(\tilde{\ell}, r)$ with $r > \tilde{r}$ in the feasibility set. This proves that the feasibility set of a lender is the strict epigraph of function h^L defined for $\tilde{\ell} > 0$ by mapping $\tilde{\ell} \mapsto \tilde{r}$. Moreover, as we showed above $h^L(0) = r^*$ and is given by (6).

To show that function h^L is increasing, fix some $r > h^L(0)$. We have established that for this r and for sufficiently small ℓ , points (ℓ, r) are in the feasibility region. Moreover, function $\Delta_i^{g+ij}(\ell, r)$ as a function of ℓ is concave, since $\partial^2 \Delta_i^{g+ij} / \partial \ell^2 = -W'_i(\ell) < 0$. It means that for any $r > h^L(0)$ there is at most one point (ℓ, r) where $\Delta_i^{g+ij}(\ell, r) = 0$. This implies that function h^L is strictly increasing. If it would not be, then it would be possible to find some $\ell_1 < \ell_2$ with $h^L(\ell_1) \geq h^L(\ell_2) > h^L(0)$. But then, for any $\hat{r} \in [h^L(\ell_2), h^L(\ell_1)]$ there would be at least two values of ℓ where $\Delta_i^{g+ij}(\ell, r) = 0$.

The proof of the second statement of this lemma, for a borrower, is similar. \square

The results of Lemma 2 are illustrated in the left panel of Fig. 1. In this example, whose parameters and other details will be discussed in Section 3.7, we depict, in the coordinates (ℓ, r) the feasibility sets of a lender (light red) and of a borrower (light blue). The higher (blue) dot marks the highest interest rate which a borrower can offer, i.e., $h^B(0)$ as given by (7). The lower (red) dot marks the lowest interest rate at which a lender can accept the deal, i.e., $h^L(0)$ as given by (6). The two horizontal lines show the penalty rate r^p and the deposit rate r^d of the Central Bank.

In this example, the highest interest rate of the borrower is within the corridor $[r^d, r^p]$ set by the Central Bank. Inspecting (7) we conclude that this a general property. In this example also the lowest interest rate of the lender is within the corridor, but this is not general. Indeed, from (6) it follows that this interest rate is not less than r^d , but can be higher than r^p , when the counter-party has a high default probability.

Note also that in the example the feasibility regions of lender and borrower intersect, suggesting that these two banks should have a link in a pairwise stable equilibrium. From the properties of the boundaries of the feasibility sets as described in Lemma 2, it follows that the feasibility region will not be empty as soon as $h^B(0) > h^L(0)$. Then we obtain the following result.

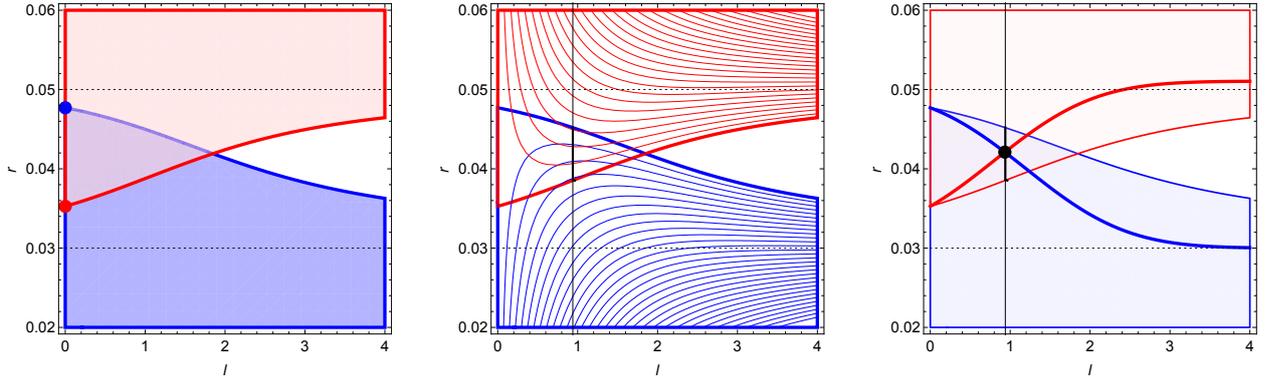


Figure 1: Illustration of Example with two banks from Section 3.7. *Left:* The feasibility sets of a lender (red) and of a borrower (blue). *Middle:* Indifference curves of lender and borrower and the contract curve (thick solid vertical line). *Right:* Supply (thick red) and demand (thick blue) curves under competitive behavior and the unique competitive pairwise stable equilibrium (black dot).

PROPOSITION 1. Consider two banks i and j with $\ell_{ij} = 0$ in the network. Loan (ℓ, r) from i to j that makes both banks better off exists if and only if

$$q_j + W_i(0) < (1 - q_j)W_j(0). \quad (8)$$

When $q_j = 0$, condition (8) becomes simply $W_i(0) < W_j(0)$ which is equivalent to $F_i(T_i - C_i) < F_j(T_j - C_j)$. For identical distributions F_i and F_j and equal reserve requirements $T_i = T_j$, this implies that the link from i to j will be created if and only if $C_i > C_j$. In case if $C_i > C_j$, introducing counter-party risk, $q_j > 0$, will increase the lower bound $h^L(0)$ for the lender, so that the intersection of feasibility sets will be shrinking.³⁶ This intersection is getting larger with a higher degree of asymmetry in the initial position of the banks. An asymmetry in reserve requirements or in the parameters of F (e.g., different variance of the shock to the predicted reserves) will also affect the region of all (ℓ, r) that improve payoffs of both banks. Higher reserve requirements will lead to higher borrowing incentives *ceteris paribus* and higher variance of the reserve shock will enhance the asymmetry.

³⁶Note that if $W_i(0) \geq W_j(0)$ (which for identical distributions and reserve requirements is equivalent to $C_i \leq C_j$), then also $q_j + W_i(0) \geq (1 - q_j)W_j(0)$, and, hence, the link will not be created *irrespective* of the level of counter-party risk.

In light of Definition 1, condition (8) has two important implications for the pairwise stable equilibrium networks. First, whenever the link from i to j exists in such network, then in the network without this link this inequality must be satisfied. Second, if the link from i to j does not exist in such network, it means that the intersection of the feasibility regions $F_{i \rightarrow j}^L$ and $F_{j \leftarrow i}^B$ is empty and, hence, the inverse inequality holds. We formulate it as

COROLLARY 1. *In every pairwise stable equilibrium network, if link from i to j does not exist then $q_j + W_i \geq (1 - q_j)W_j$.*

3.3.2 Contract Curve

For the two banks i and j which can individually gain from an ij link, the problem of finding a loan contract (ℓ_{ij}, r_{ij}) between them can be described as a bargaining problem within their feasibility set given by the intersection of $F_{i \rightarrow j}^L$ with $F_{j \leftarrow i}^L$. The second requirement of Definition 1 imposes that in the pairwise stable equilibria, the banks should find themselves in the point which is Pareto-efficient for them given the rest of the network. In other words, whenever $\ell_{ij} > 0$ in network g , the banks i and j must choose (ℓ_{ij}, r_{ij}) in such a way that no Pareto-efficient deviation from this point is possible. The set of such points within the feasibility set is called *contract curve*.

The contract curve in the coordinates (ℓ_{ij}, r_{ij}) is characterized by the condition that the slopes of indifference curves of the two banks coincide. Therefore it holds that

$$-\frac{\partial \pi_i / \partial \pi_i}{\partial \ell_{ij} / \partial r_{ij}} = -\frac{\partial \pi_j / \partial \pi_j}{\partial \ell_{ij} / \partial r_{ij}}. \quad (9)$$

With a bit of computations (see Appendix 3.10) we find that this condition becomes

$$q_j + r^d + (r^p - r^d)F_i(T_i - C_i) = (1 - q_j) \left(r^d + (r^p - r^d)F_j(T_j - C_j) \right),$$

or, using the expected marginal rate notation introduced in (5), simply

$$q_j + W_i = (1 - q_j)W_j. \quad (10)$$

This condition, that should be satisfied in the pairwise stable equilibrium for every existing link, relates the quantities W_i and W_j for every two pair of lender and borrower. Note that the expected marginal rates, W 's, do not depend on interest rate r_{ij} . They both depend on ℓ_{ij} , and the equation (10), hence, can be thought of as an equation to determine ℓ_{ij} given the rest of the network. Next result assures that if such ℓ_{ij} exists, then, it is unique, and moreover established a necessary and sufficient condition for an existence of (ℓ_{ij}, r_{ij}) on the contract curve.

PROPOSITION 2. *Consider an arbitrary network g , and any two banks i and j with non-empty intersection of feasibility sets $F_{i \rightarrow j}^L$ with $F_{j \leftarrow i}^B$ and such that $ij \notin g$. Then there exists a unique $\ell_{ij} > 0$ that is consistent with requirement 2 from Definition 1 of the pairwise stable equilibrium.*

Proof. When an intersection of the feasibility sets is not empty, Proposition 1 implies that $q_j + W_i(0) < (1 - q_j)W_j(0)$. When ℓ increases, the left hand-side strictly increases and the right hand-side strictly decreases. Lemma 1 implies that when $\ell \rightarrow \infty$, the left hand-side approaches $q_j + r^p$, whereas the right hand-side approaches $(1 - q_j)r^d$. Since $q_j + r^p \geq r^p > r^d > (1 - q_j)r^d$, there exists a unique $\ell_{ij} > 0$ that solves equation (10).

We now have to show that for this ℓ_{ij} there is at least one interest rate r such that (ℓ_{ij}, r) are in the intersection of the feasibility sets. Consider $r_{ij} = (q_j + W_i)/(1 - q_j) = W_j$, where the last equality is due to (10). It is easy to check that in the point (ℓ_{ij}, r_{ij}) the indifference curves of both lender and borrower are horizontal. From Lemma 2 we know that the indifference curve corresponding to $\Delta_i^{g+ij}(\ell, r) = 0$ is given by function $h^L(\ell)$ and so its slope in ℓ_{ij} is positive. It is easy to see that the slope of indifference curve of lender for a given ℓ decreases with r . Therefore, $r_{ij} > h^L(\ell_{ij})$, which means that $(\ell_{ij}, r_{ij}) \in F_{i \rightarrow j}^L$. In a similar way, we can show that $(\ell_{ij}, r_{ij}) \in F_{j \leftarrow i}^B$. \square

The central panel of Fig. 1 illustrates the previous result. We show there several indifference curves of both lender (red) and borrower (blue) belonging to their feasibility sets (i.e., these are isoprofit curves when profits are larger than in the empty network). The vertical

black line represents all the points in coordinates (ℓ, r) for which Eq. (9) is satisfied. In this example, the intersection of the feasibility sets is not empty and so the contract curve exists as shown by the thick vertical line. There is a whole interval of interest rates that are consistent with the pairwise stable equilibrium network.

Proposition 2 immediately implies the following

COROLLARY 2. *In every pairwise stable network, for any two banks with i and j with a link $\ell_{ij} > 0$, it holds that $q_j + W_i = (1 - q_j)W_j$.*

3.4 Equilibrium Network Configurations

We now apply the results from the previous section to characterize the networks that can emerge in the pairwise stable equilibrium. As the discussion above implies, there might be infinitely many such equilibria, because requirements of Definition 1 are too weak to fix the interest rate. Some extra assumptions are needed and we will discuss them in the next section. On the other hand, as we will show now, Definition 1 is sufficient not only to give quite precise predictions about the loan amounts, given the values of the exogenous variables, but also describe which types of networks should in general be observed in the pairwise stable equilibrium.

Our first result shows that a pairwise stable equilibria cannot have pair of banks that lend to each other.

LEMMA 3. *In a pairwise stable equilibrium, if $\ell_{ij} > 0$ for a pair of banks, i and j , and if either $q_i > 0$ or $q_j > 0$, then $\ell_{ji} = 0$.*

Proof. Suppose the contrary, i.e., that i and j lend funds to each other. Consider bank i . As it is both lender and borrower for j , Corollary 2, implies that

$$W_i = (1 - q_j)W_j - q_j = (1 - q_j)(1 - q_i)W_i - (1 - q_j)q_i - q_j < (1 - q_j)(1 - q_i)W_i,$$

where in the last inequality we used the fact that at least one of the two banks have non-

zero probability of default. Clearly the last inequality implies that $W_i < W_i$, which is a contradiction. \square

The second result shows that at every equilibrium, we cannot have directed loops.

LEMMA 4. *In a pairwise stable equilibrium, if we have a finite set of banks $\{i_1, i_2, \dots, i_k\}$, with $q_j > 0$ for at least one $j \in \{i_1, i_2, \dots, i_k\}$, such that $\ell_{i_h i_{h+1}} > 0$ for any consecutive pair of banks, then $\ell_{i_k i_1} = 0$.*

Proof. From Corollary 2 we have that in such pairwise stable equilibrium network for any $1 \leq h \leq k$ it is

$$W_{i_h} = (1 - q_{i_{h+1}})W_{i_{h+1}} - q_{i_{h+1}} \leq W_{i_{h+1}}.$$

Suppose the contrary, i.e., that $\ell_{i_k i_1} > 0$ in the equilibrium. Then $W_{i_k} \leq W_{i_1}$ and generally

$$W_{i_k} \leq W_{i_1} \leq W_{i_2} \leq \dots \leq W_{i_k}$$

with at least one strict inequality. But this is impossible. \square

The third result shows that there are no directed paths of length greater than one or, in other words, there is no intermediary.

LEMMA 5. *In a pairwise stable equilibrium, we cannot have three banks $\{i, j, k\}$, with $q_j > 0$, such that $\ell_{ij} > 0$ and $\ell_{jk} > 0$.*

Proof. Assume the contrary, i.e., there are three banks with $\ell_{ij} > 0$ and $\ell_{jk} > 0$. First, we will show that then it also must be the case that $\ell_{ik} > 0$. Indeed, from Corollary 2 we have that

$$W_i = (1 - q_j)W_j - q_j < W_j = (1 - q_k)W_k - q_k.$$

This means that $q_k + W_i < (1 - q_k)W_k$ and from Corollary 1 there must be a positive loan from i to k in the equilibrium.

But then from Corollary 2 we have that

$$W_i = (1 - q_k)W_k - q_k = W_j,$$

where the first equality is because i lends to k and the second equality is because j lends to k . Therefore, $W_i = W_j$, which is impossible given that i lends to j with $q_j > 0$. \square

The fourth result shows that there will generically be no separated components in the pairwise stable equilibrium network.

LEMMA 6. *In a pairwise stable equilibrium, we cannot have four banks $\{i, j, k, h\}$, such that $\ell_{ij} > 0$, $\ell_{kh} > 0$, and $W_i \neq W_k$.*

Proof. Suppose the contrary, that is for those four banks $\ell_{ij} > 0$, $\ell_{kh} > 0$, and, say, $W_i > W_k$. From Corollary 2, for the link from i to j we have that

$$(1 - q_j)W_j - q_j = W_i.$$

Therefore, $W_k < (1 - q_j)W_j - q_j$, and Corollary 1 implies that then it must be $\ell_{kj} > 0$. But then, along this link, applying Corollary 2, we have that $W_k = (1 - q_j)W_j - q_j$, which is a contradiction.

If $W_k > W_i$, then similar reasoning applies to the links to h . \square

All four lemmas of this section imply the following result.

PROPOSITION 3. *A pairwise stable equilibrium is such that banks are partitioned in three groups: isolated banks, borrowers and lenders. Borrowers and lenders form generically a unique component where all directed paths have at most length one.*

3.5 Interest rate under competitive behavior

Corollaries 1 and 2 give an analytical characterization for every existing and non-existing link in the pairwise stable networks. Condition (10), allows one to determine the loan

amounts in a unique way for every existing link, given the rest of the network. At the same time, the notion of pairwise stable networks is too weak to specify interest rates precisely, though it imposes certain boundary conditions on them. The next notion is a possible refinement of the pairwise stable equilibria that leads to more strict predictions about the interest rates observed in the network.

DEFINITION 2. *Network $g = (\mathbf{L}, \mathbf{r})$ is a **competitive pairwise stable equilibrium** if it is the pairwise stable equilibrium and on every existing link ij , the amount $\ell_{ij} > 0$ maximises the profits of both lender i and borrower j given the interest rate r_{ij} .*

In a competitive pairwise stable equilibrium the banks exhibit the price-taking behavior in their bilateral interactions. This assumption may be realistic when the number of banks is large and when there is approximately as many potential lenders (banks expecting high s 's) and potential borrowers (banks expecting low s 's).

To study the network in the competitive pairwise stable equilibrium, let us assume, as before, that given the rest of the network, bank i considers lending some reserves to bank j . For lending bank i , differentiating π_i with respect to ℓ_{ij} , we find that the first order-condition is given by

$$r_{ij}(1 - q_j) = q_j + r^d + (r^p - r^d) F_i(T_i - C_i). \quad (11)$$

Moreover, the second-order condition guarantees that the solution of this equation delivers the maximum profit to a lender, given the interest rate r_{ij} .

When bank i decides to lend to bank j , it compares the marginal cost of this decision given by the right hand-side of (11) with the marginal benefit of it given by the left hand-side of (11). The marginal cost are equal to the probability of default of the borrower (as then the principal is lost) plus the marginal increase in the expected cost of dealing with the Central Bank. The marginal benefit is the expected marginal payment from bank j . Note that the interest rate consistent with the lender's first-order condition is always above r^d but may also be above r^p . This is a consequence of a risk that bank j may default.

Note that (11) can be written as $r_{ij}(1 - q_j) = q_j + W_i(\ell_{ij})$. This implicitly defines the

supply function of lender i for the funds to borrower j . Since $W'_i > 0$, the supply function is strictly increasing.

Consider now bank j who is borrowing from i . Differentiating π_j with respect to ℓ_{ij} , we find that the first order-condition for j is given by

$$r_{ij} = r^d + (r^p - r^d) F_j(T_j - C_j). \quad (12)$$

The second-order condition is again satisfied. When bank j decides to borrow, it compares the marginal cost of this decision given by the left hand-side of (12) with the marginal benefit of it given by the right hand side of (12). The marginal cost of borrowing is given by the interest rate r_{ij} . The marginal benefit is the marginal decrease in the expected cost of dealing with the Central Bank.

Note that the rate that borrower j pays does not depend on the identity of lender, but only on the lending amount, according to (12). Therefore, in the competitive pairwise stable equilibrium, there will be only one interest rate per borrower, which we call this borrower's interest rate and denote as r_j . Note also that this interest rate is always between r^d and r^p .

Rewriting (12) as $r_{ij} = W_j(\ell_{ij})$ defines implicitly the demand function for the borrower j . This function is decreasing as $W'_j < 0$.

Combining (11) and (12) we obtain that in the competitive pairwise stable equilibrium it must be that

$$r_j = \frac{q_j + W_i(\ell_{ij})}{1 - q_j} = W_j(\ell_{ij})$$

The second equality is equivalent to (10), confirming that competitive behavior is consistent with Pareto efficiency. In fact, this result shows that as soon as in the network the contract curve for banks i and j is not empty, there exists a unique pair of (ℓ_{ij}, r_{ij}) that satisfies the first order conditions of both lender and borrower. Moreover, as we showed in the proof of Proposition 2, this point (ℓ_{ij}, r_j) belongs to the feasibility regions of both buyer and seller. In this sense the notion of competitive pairwise stable equilibrium refines the pairwise stable equilibrium from Definition 1.

We illustrate the competitive pairwise stable equilibrium in the right panel of Fig. 1, where the thick red and blues curves show, respectively, the supply of loans and demand for loans schedules as derived from (11) and (12). The black point gives the equilibrium values of ℓ and r .

3.6 Monetary policy and market conditions

The Central Bank may influence the interbank market interest rate and volumes through a range of monetary policy instruments. In turn, banks pass through the interest rate changes to the rest of the economy. In our model the policy variables are: deposit rate r^d , penalty rate r^p , and bank-specific minimum reserve requirements T 's. Indirectly through open market operations (buying/selling government bonds, lending through repurchase agreements), the Central Bank may influence the overall liquidity and level of bank reserves in the economy.³⁷ As we showed in Section 3.5, under competitive behavior the interest rate of the borrower is always between r^d and r^p , and the interest rate of the lender is above r^d , but may be above r^p . It is easy to see that r^d and r^p affect the marginal rate of banks transaction with the Central Bank, W 's as defined in (5). Hence, increasing r^d or r^p will increase W 's and the average overnight market interest rate, *ceteris paribus*. Intuitively increase in the Central Bank deposit rate r^d , will shift the lender's supply curve up as their outside option of depositing money with the bank will improve. On the other hand, an increase in r^p will result in an upward shift of the demand curve from the borrowers as they will face a larger penalty rate. Similarly, minimum reserve requirements T_i 's are positively related to W_i 's and their increase will result in higher average overnight market rate. Finally, open market operations target the overall liquidity in the system and thus affect s_i 's. Since W_i is negatively related with s_i , an increase of the overall liquidity will decrease the average overnight interest rate.

Market conditions may also have an effect on the overnight market rate and transaction

³⁷The range of used policy instruments differs from country to country. For instance, the ECB employs all mentioned instruments. Until 6 October 2008, the US Fed paid zero deposit rate on excess reserves and introduced positive deposit rate as an anti-crisis measure. The People's Bank of China actively uses the minimum reserves requirements to limit inflation. Canada, the UK, New Zealand, Australia and Sweden have no reserve requirements and instead use capital requirements.

volumes. When market conditions worsen, the probabilities of default, q_i , may increase. This will shift the supply curve of the lenders up as their expected marginal payoff will decrease (when the increased defaults are factored in). The demand curve of the borrowers will not be affected. This will result in a higher equilibrium overnight market rate and smaller number of transactions. For a sufficiently high default probability, the supply curve may shift above the demand curve for all r 's and result in “market freeze” with zero transaction volumes and undefined overnight market rate.

Interestingly, increased uncertainty about the reserves, σ , has different implications for lenders and borrowers. Specifically, an increase in σ results in an increase in W in the case when $T > C$ and in a decrease in W when $C < T$. This implies that the demand curve of borrowers will shift downwards and the supply curve of lenders will shift upwards, resulting in smaller transaction volumes.

3.7 Examples

We consider here few numerical examples to illustrate the concepts and results presented so far. In all examples we will assume for the sake of simplicity, that all banks are homogeneous in all respects except for their initial predictions of their cash positions. Specifically, banks are subjects to the funds' uncertainty shocks generated from the normal CDF with mean 0 and variance $\sigma^2 = 1$. All banks have probability of default $q = 0.001$. Parameters of the Central Bank are: deposit rate $r^d = 0.03$, penalty rate $r^p = 0.05$, the reserve requirements common for all banks are $T = 2.2$.

There will be four banks overall, A, B, C and D, but we start with situations when only some of these banks are present. Banks A and C are identical and relatively good in terms of projections of their reserves, for these banks $s_A = s_C = 3$. With $T = 2.2$ it means that the banks will not meet the reserve requirements with probability $F(2.2 - 3) \approx 0.21$. Their profit for a given parameters in the absence of interbank market is approximately 0.022 according to (2). Banks B and D expect to have fewer reserves, $s_B = 1$ and $s_D = 0.9$. Their probabilities of not meeting the requirements are, respectively, $F(2.2 - 1) \approx 0.88$ and $F(2.2 - 1.1) \approx 0.82$,

whereas their profits in the absence of interbank market are approximately -0.061 and -0.066 , respectively. We give these benchmark values for profit in the first row of Table 20. In the same table, for the examples discussed below, when specific banks are present, we identify all the networks consistent with pairwise stable equilibria and report the corresponding loans. We also find the competitive pairwise stable equilibria and report the interest rates and the corresponding banks profits (and below, in parentheses, the profit difference with respect to the empty network). Finally, for the equilibrium networks we compute the total volume of trade in the interbank market (4th column) and the expected loss of funds for the interbank market due to exogenous default (the last column) defined as

$$\text{ELoss} = \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{N}} \ell_{ik} q_k.$$

Two banks. Let us consider the case with only two banks A and B. In this case we can expect that bank A may find profitable to lend some funds to bank B but not *vice versa*. Fig. 1 illustrates the feasibility regions, contract curve and competitive pairwise stable equilibrium for these two banks with bank A as a lender and bank B as a borrower. In this case, there is a unique loan amount ℓ_{AB}^* in the pairwise stable equilibrium and a non-empty range of possible interest rates $[r_{AB}^l, r_{AB}^u]$. When the banks are behaving competitively, there is a unique interest rate $r_{AB}^* \in [r_{AB}^l, r_{AB}^u]$. The values of variables³⁸ ℓ_{AB}^* and r_{AB}^* can be found in the second row of Table 20. There are no any other pairwise stable equilibrium. Indeed, the empty network is not an equilibrium, as follows from Corollary 1 (and illustration in Fig. 1). Adding the link from B to A to the existing configuration cannot lead to the equilibrium (Lemma 3). Finally, there is no pairwise stable equilibrium with only one loan from B to A, as it follows from Proposition 1 (see also footnote 36).

³⁸For this example with two banks, we can also compute $r_{AB}^l \approx 0.039$, $r_{AB}^u \approx 0.045$. However, with more banks reporting the interval of interest rates consistent with the bilaterally stable equilibrium is virtually impossible. Indeed, the interval along any existing link depends on the loan amount *and* interest rates on all other existing links.

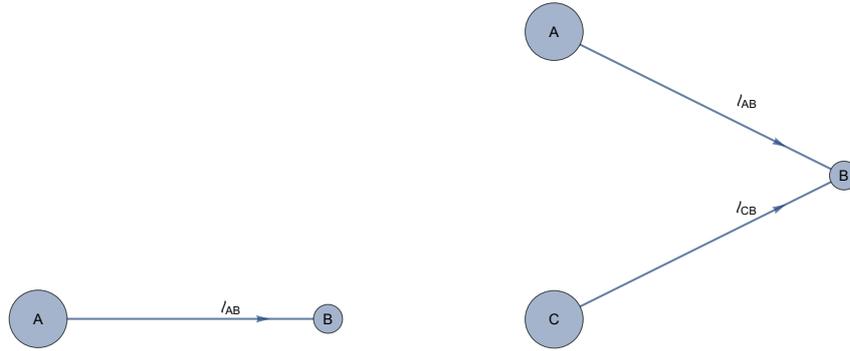


Figure 2: Networks of loans that can be observed in the pairwise stable equilibria in the examples of Section 3.7 with two and three banks.

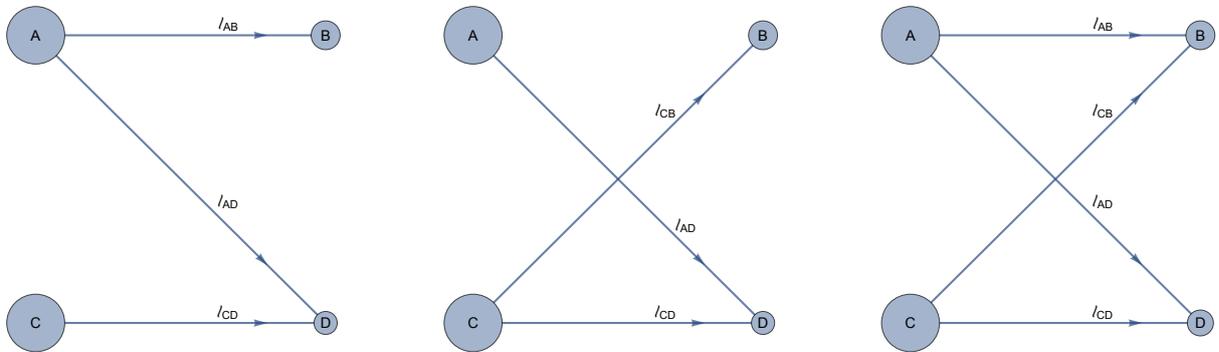


Figure 3: Networks of loans that can be observed in the pairwise stable equilibria in the examples of Section 3.7 with four banks.

Three banks. Let us add bank C which is the same as A, $s_C = 3$. It turns out that now the configuration of the previous equilibrium (i.e., with the only link from A to B) is not an equilibrium. This is because C, being not connected but with relatively large reserves, has a low expected marginal rate W_C and can find a profitable deal with bank B. Numerical computations shows that there is only one pairwise stable equilibrium in this case. In this equilibrium banks A and C both lend to B (see the right panel in Fig. 2 and the third row of Table 20). In comparison with the case of two banks, bank B has a higher profit, but bank A has a lower profit, even if it is able now to keep higher reserves. In a sense, even with competitive behavior, an existence of bank C undermines possibility of A to lend at a higher interest rate and get a higher expected profit from the interbank market.

Four banks. We now add bank D which is similar to B, in the sense that it is also a bank with relatively low reserves, though it expects even lower reserves than B, $s_D = 0.9$. The equilibrium configuration of the previous case (A and C lend to B) is not a pairwise stable equilibrium any longer because every of the existing banks and bank D could make a profitable link. The configuration when A and C lend to D is not an equilibrium for the same reason. In this configuration even bank D is willing to lend money to B, even if *ex ante* bank D expects to have less reserves.

Numerically we find that there are three network configurations consistent with pairwise stable equilibrium: (i) A and C lend to D and A lends to B, (ii) A and C lend to D and C lends to B, and (iii) both A and C lend to B and D (see Fig. 3 and the last three rows of Table 20). However, all these equilibria are payoff equivalent for all banks.

3.8 Data

In this Section we relate our theoretical findings to the data from an overnight interbank market. In particular, we analyse the Electronic Market for Interbank Deposit, e-MID, based in Milan. This is one of the dominant interbank unsecured deposit market on an electronic platform for the Euro area.³⁹ The e-MID is open to all European banks and European branches of non-European banks. We have access to the e-MID transaction data for the period January 1999 – September 2009. During this period, 350 banks participated in this electronic market. The most active segment of the market consists of the overnight lending.

The market is organised in the form of an electronic book, where offers can be submitted and stored. Any bank may post the offer specifying the bid (for borrowing funds) or ask (for lending funds) side, maturity term, rate and amount. Typically the identity of such bank, *the quoter*, is revealed. Best quotes, i.e., the highest bid and the lowest ask, are displayed at the top of the book, with other bids displayed in descending order and other asks displayed in ascending order in the respective side of the book. Any bank may be *an*

³⁹Based on “Euro Money Market Study 2006” by ECB, e-MID accounts for 17% of turnover in unsecured money market in the Euro Area. A more recent 2010 study indicates reduction in the total turnover to 10%.

Example	Configuration	Loan	Volume	Interim Reserves			Interest rate	Profit			Exp Loss	
				C_A	C_B	C_C		C_D	π_A	π_B		π_C
	empty network	0	0	3	1	3	0.9	0.0216	-0.0611	0.0216	-0.0659	0
two banks A and B	A lends to B	$\ell_{AB} = 0.933$	0.933	2.067	1.933			$r_{AB} = 0.042$ $(+0.0033)$	-0.0582 $(+0.0029)$			0.00093
three banks A, B and C	A lends to B	$\ell_{AB} = 0.623$	1.2459	2.377	2.2458	2.377		$r_{AB} = 0.0396$	-0.0555	0.0230		0.0012
	C lends to B	$\ell_{CB} = 0.623$						$r_{CB} = 0.0396$	$(+0.0014)$	$(+0.0014)$		
four banks - 1 A, B, C and D	A lends to B	$\ell_{AB} = 0.9080$	1.9159	2.0420	1.9080	2.0420	1.9080	$r_{AB} = 0.0423$	-0.0584	0.0251	-0.0626	0.0019
	A lends to D	$\ell_{AD} = 0.0500$						$r_{AD} = 0.0423$	$(+0.0035)$	$(+0.0035)$	$(+0.0035)$	
four banks - 2 A, B, C and D	A lends to D	$\ell_{AD} = 0.9580$	1.9159	2.0420	1.9080	2.0420	1.9080	$r_{AD} = 0.0423$	-0.0584	0.0251	-0.0626	0.0019
	C lends to B	$\ell_{CB} = 0.9080$						$r_{CB} = 0.0423$	$(+0.0035)$	$(+0.0035)$	$(+0.0035)$	
four banks - 3 A, B, C and D	A lends to B	$\ell_{AB} = 0.6066$	1.9159	2.0420	1.9080	2.0420	1.9080	$r_{AD} = 0.0423$	-0.0584	0.0251	-0.0626	0.0019
	C lends to B	$\ell_{CB} = 0.3014$						$r_{CB} = 0.0423$	$(+0.0035)$	$(+0.0035)$	$(+0.0035)$	
	A lends to D	$\ell_{AD} = 0.3514$	1.9159	2.0420	1.9080	2.0420	1.9080	$r_{AD} = 0.0423$	-0.0584	0.0251	-0.0626	0.0019
	C lends to D	$\ell_{CD} = 0.6566$						$r_{CD} = 0.0423$	$(+0.0035)$	$(+0.0035)$	$(+0.0035)$	

Table 20: Network configurations in the pairwise stable equilibria and in the competitive pairwise stable equilibria for examples with two, three and four banks.

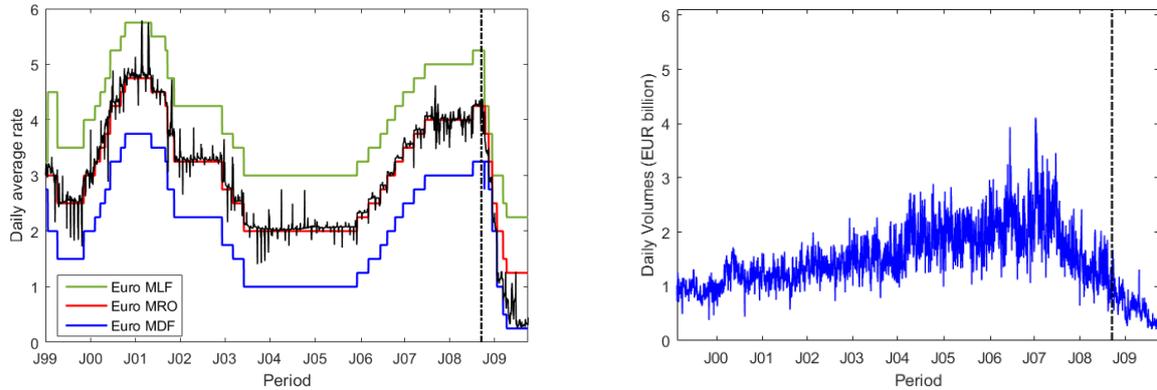


Figure 4: *Left panel:* Daily volume-weighted average e-MID rates (black line) vs the key ECB rates: Euro Marginal Deposit Facility (MLF), Euro Marginal Lending Facility and (MLF) and Euro Main Refinancing Operation (MRO) rates. *Right panel:* daily trading volumes. Dotted black line indicates Lehman Brothers default date (15 September 2008). The labels on the x-axis correspond to the first trading day of January for a given year.

aggressor, which means that it may pick a quote from the book, negotiate some terms of the offer with the quoter and accept it. The quoter has an option to reject the aggressor’s terms after negotiation and after revealing the aggressor’s identity. Once the transaction is executed and the book is updated, the transaction terms (maturity, rate, volume) and time stamp are revealed to all market participants.

There is a number of empirical studies using e-MID data. ? finds that on daily frequency preferential transactions are limited in the sense that degree distribution does not exhibit fat tails. Moreover, there are virtually no intermediaries. These findings are consistent with our theoretical model. However, for longer aggregation periods (e.g., on monthly, quarterly and annual frequencies) some banks in the e-MID trading network exhibit much higher concentration of links than others, leading to the fat-tails in the degree distribution. Moreover, core-periphery type structures emerge (Finger, Fricke, and Lux, 2013; Fricke and Lux, 2014). ? suggest a trading model with memory that supports preferential attachment and leads to the fat-tailed distribution.

As our network formation model is best suited to describe day patterns we will focus on the daily frequency analysis. The left panel of Fig. 4 shows daily volume-weighted average

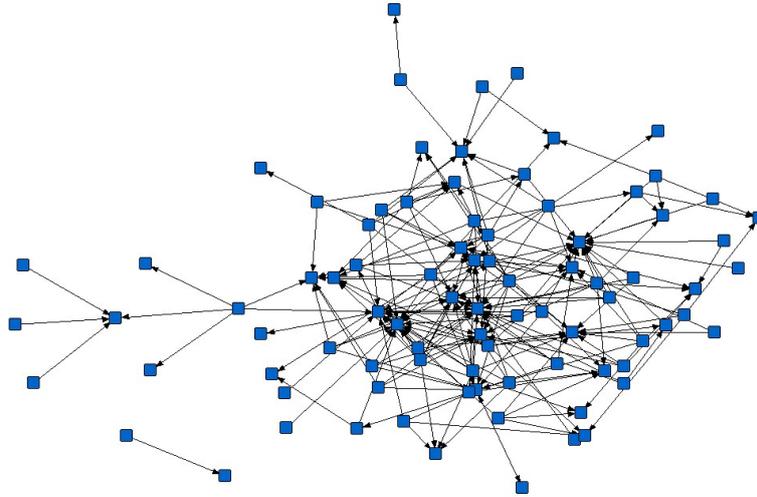


Figure 5: Network of lending and borrowing on the e-MID market. Aggregate transactions for one day. Arrows indicate direction of the loan

rates on e-MID overnight loans (black line with spikes) along with the key ECB rates. The upper line is the Euro Marginal Lending Facility, the penalty rate at which banks can borrow from the ECB overnight. The lower line is the Euro Marginal Deposit Facility rate at which banks deposit excess reserves with the ECB overnight. Finally, there is a line in between (shown in red) that is the Euro Main Refinancing Operations (MRO) rate at which banks may borrow for one week by providing acceptable collateral to the ECB.

Consistently with the results of our model, we observe that the e-MID rates are between the ECB deposit and lending (penalty) rate, and on average are close to the ECB MRO rate. There is some volatility observed in the average daily rate. Similarly the daily trading volumes (see the right panel of Fig. 4) exhibit substantial variation. In part volatility in the rate and volumes is due to the fact that banks in Euro area are required to meet a minimum reserve requirements on a certain day in the end of the maintenance period. The vertical dotted line indicates 15 September 2008, the date of default of Lehman Brothers. The crisis period after this episode is characterised by small trading volumes and excess reserves as the ECB was providing extra liquidity in the system. Our model predicts that excess liquidity in the system will drive the interest rates down to the Central bank deposit rate, which is

Year	Banks	Var(r_{len})	Var(r_{bor})	Prop 2 loops	Prop 3 loops
1999	215	0.0030	0.0027	0.0140	0.0124
2000	196	0.0034	0.0025	0.0105	0.0079
2001	185	0.0037	0.0028	0.0163	0.0087
2002	177	0.0032	0.0024	0.0205	0.0077
2003	179	0.0031	0.0023	0.0174	0.0095
2004	180	0.0029	0.0022	0.0145	0.0064
2005	176	0.0029	0.0021	0.0175	0.0083
2006	177	0.0027	0.0020	0.0133	0.0074
2007	178	0.0029	0.0021	0.0095	0.0035
2008	173	0.0034	0.0024	0.0058	0.0016
2009	153	0.0035	0.0024	0.0043	0.0000
Overall	350	0.0032	0.0024	0.0131	0.0067

Table 21: Characteristics of the e-MID market. For each year column 2 shows the number of active banks on the market; columns 3 and 4 report the volume-weighted average variance of daily individual lending and borrowing rate, respectively; columns 5 and 6 report the proportion of banks involved in 2-loop cycle or 3-loop cycle to the total number of active banks.

consistent with observations in this period.

Fig. 5 illustrates the trading network for a typical trading day (14 January 2009). Arrows indicate the direction of funds flow (from a lender to a borrower). As our model suggests we observe the network with a small number of intermediaries. Remarkably most of the active banks are connected in one giant components. Only two banks are disconnected from the rest of the network. Existence of one giant component including nearly all active banks is a typical daily pattern consistently observed in the data over all considered years. A similar pattern is also predicted by our theoretical model where lenders/borrowers tend to transact with multiple counter-parties.

Our theoretical model predicted that in the competitive pairwise stable equilibrium any borrowing bank will borrow at the same rate from multiple lenders, while lending banks may differentiate the interest rates between different borrowers. To verify whether this holds in the e-MID market we proceed as follows. We identify lenders and borrowers who transacted with more than one counter-party during one day. Then, we calculate for each such bank,

the variance of their lending and borrowing rates, respectively. We take a volume-weighted average of these variances over all identified lenders and borrowers and over the number of days for each year. The results are reported in Table 21, columns 3 and 4. The weighted average variance of the identified lenders rates is higher than the weighted average variance of the identified borrowers rate. This result is consistently for all the considered years. Our model predicts that the borrowers' variance should be equal to zero in a short period, but in practice this is not holding exactly as the borrowers liquidity needs may change over the day and this will trigger changes in demand and, hence, different borrowing rates. Columns 5 and 6 of Table 21 report the proportion of banks involved in two and three cycles. Our model rules out cycles (even in the absence of competitive behavior) and the data confirm that the proportion of banks involved in cycles is negligible. Again, the cycles may arise due to liquidity changes, but not because of strategic considerations.

3.9 Conclusions

In this paper we built a model of endogenous network formation in the interbank market, with lending banks facing the trade-off between an uncertain gain on the loaned funds and a higher possibility of not meeting reserve requirements. We find that the concept of pairwise stability is sufficient to identify certain network structures which are reminiscent to those that emerge on daily frequency in the real market for interbank lending. Namely, our model predicts the emergence of bipartite network, where lenders and borrowers are connected in one giant component. There are no cycles and intermediaries in the network.

Our model produces intuitive responses to changes in the monetary policy instruments and varying market condition. In particular the model predicts that an increase in the Central Bank lending or borrowing rate or the minimum reserves requirements will increase the average overnight market rate. Larger provability of defaults will increase the average market rate, reduce the volumes and may lead to the “market freeze” where no trade is feasible. The increase in the uncertainty about daily reserves will lower the transition volumes.

In the further analysis, we may consider a stricter version of network equilibrium con-

figurations than the stability concept we employed here. For instance, banks may start to reconsider several links at the same time and they also can suggest doing the same to their counter-parties. This refinement is similar to the concept of bilateral stability studied in a different context by Goyal and Vega-Redondo (2007). Furthermore, banks may not behave myopically, but instead foresee which further changes to the network structure will follow their particular decision to establish or sever a link, and in this case one could for example adapt the framework of Herings, Mauleon, and Vannetelbosch (2009).

There are many other possible directions in which the model can be generalized to be more realistic. First, one can relax an assumption of risk neutrality of banks (i.e., expected profit maximization), which rules out any incentive of banks for diversification in their lending behavior. When banks become sensitive to risk, lenders who face a counter-party default risk will be more willing to diversify their supply of lending funds through various banks. In such a model, it would be interesting to study the case when probabilities of default of different banks are not independent. Second, one can introduce further effect to the interbank market by letting default probabilities be affected by the decision in the interbank market. This would not change the borrower's incentives but would make lenders' tasks more complicated introducing nonlinearities in their behavior.

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APPENDIX

3.10 Derivatives, First and Second order conditions

We compute the derivatives of π_i given in (4) with respect to the borrowing quantity ℓ_{ji} from bank j and also with respect to the lending quantity ℓ_{ij} to bank j .

Derivatives for the borrower. When bank i wants to borrow from bank j , its marginal profit is

$$\begin{aligned} \frac{\partial \pi_i}{\partial \ell_{ji}} &= -r_{ji} + r^d + F_i(T_i - C_i)(r^p - r^d) - (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) + \\ &\quad + (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) = \\ &= -r_{ji} + r^d + (r^p - r^d) F_i \left(T_i - s_i + \sum_k \ell_{ik} - \sum_m \ell_{mi} \right) \end{aligned} \quad (13)$$

Setting this to zero we get the first-order condition for the borrower (12).

The second derivative reads

$$\frac{\partial^2 \pi_i}{\partial \ell_{ji}^2} = -(r^p - r^d) f_i(T_i - C_i) < 0.$$

The derivative of profit by the interest rate is

$$\frac{\partial \pi_i}{\partial r_{ji}} = -\ell_{ji}.$$

When we impose that bank i lends money to bank j then the computations above imply that the slope of indifference curve of borrower j in the coordinates (ℓ_{ij}, r_{ij}) is given by

$$-\frac{\partial \pi_j / \partial \ell_{ij}}{\partial \pi_j / \partial r_{ij}} = \frac{-r_{ij} + r^d + (r^p - r^d) F_j(T_j - C_j)}{\ell_{ij}}.$$

This result is then substituted to the right hand-side of condition (9) in order to get the result (10) in the main text.

Derivatives for the lender. When bank i wants to lend to bank j , its marginal profit is

$$\begin{aligned}
\frac{\partial \pi_i}{\partial \ell_{ij}} &= r_{ij}(1 - q_j) - q_j - r^d - F_i(T_i - C_i)(r^p - r^d) + (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) - \\
&\quad - (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) = \\
&= r_{ij}(1 - q_j) - q_j - r^d - (r^p - r^d) F_i \left(T_i - s_i + \sum_k \ell_{ik} - \sum_m \ell_{mi} \right)
\end{aligned} \tag{14}$$

Setting this to zero we get the first-order condition for the lender (11). The second-order derivative is given by

$$\frac{\partial^2 \pi_i}{\partial \ell_{ij}^2} = -(r^p - r^d) f_i(T_i - C_i) < 0$$

which is the same as for the borrower.

The derivative of profit by the interest rate is

$$\frac{\partial \pi_i}{\partial r_{ij}} = \ell_{ij}(1 - q_j).$$

When we impose that bank i lends money to bank j then the computations above imply that the slope of indifference curve of lender i in the coordinates (ℓ_{ij}, r_{ij}) is given by

$$-\frac{\partial \pi_i / \partial \ell_{ij}}{\partial \pi_i / \partial r_{ij}} = -\frac{r_{ij}(1 - q_j) - q_j - r^d - (r^p - r^d) F_i(T_i - C_i)}{\ell_{ij}(1 - q_j)}.$$

This result is then substituted to the left hand-side of condition (9) in order to get the result (10) in the main text.