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The usual disclaimer applies.

Contents

List of Figures

List of Tables

Introductory Chapter

This chapter is a brief introduction to the content of this dissertation. The following chapters, although seem independent, are all written in the classical tradition of Sraffa's revival, and are all inspired by the contributions of Professor Salvadori (many of these contributions are co-authored with Professor Kurz) in this literature, to whom I own my deepest debt. These chapters concern the problems of fixed capital, rent and exhaustible resources.

Chapter 1 is a survey on the development of fixed capital models in the Sraffa framework. Sraffa (1960) proposed a fixed capital model with a single machine which has constant efficiency. His model was widely generalised, and these generalisation can be roughly categorised according to two criteria of the properties of old machines: transferability and joint utilisation. The former criterion concerns whether or not an old machine can be jointly utilised with other old machines in the same production process, and the latter concerns whether or not an old machine is used in different sectors. Much of the literature is devoted to the models that exclude the transferability of old machines. This is because of the fact that the transferability may cause some complexities of those in the pure joint production. However, as Sraffa suggested, this worry is not necessary if machines always work with constant efficiencies. Actually, if machines always work with constant efficiencies, there exists no need to assume non-transferability nor non-jointly utilisation of old machines. The above two criteria only matter when machines have variable efficiencies. Hence, Sraffa's model may not be as special as it appears. After illustrating some common assumptions, this survey then proceeds to the models with machines of constant efficiencies. In what follows, I discuss the models with machines of variable efficiencies based on the above two criteria. Attention is focused on the properties of the cost-minimising technique. This chapter, served as a general introduction to topic of fixed capital, is followed by Chapter 2, which is devoted to a special fixed capital model, a model with both transferable and jointly utilised machines.

As can be seen from Chapter 1, much of the literature in the fixed capital models excludes transferable old machines. Developing the suggestions given by Sraffa, Salvadori (1999) builds a model with transferable non-jointly utilised machines whose efficiencies are not constant, but are still independent of the sectors in which the machines are used (an assumption called the Uniform Efficiency Path). He shows that the neat properties of the non-transferable single machine model still hold. Chapter 2 follows Salvadori's suggestions in his paper "I conjecture that a similar formalism can be found for the case in which machines are used jointly" (Salvadori 1999, p. 270), and seeks to study the model with both transferable and jointly utilised machines by applying the Uniform Efficiency Path Axiom, that is, to synthesise the model with non-transferable jointly utilised machines (Salvadori 1988) and the one with transferable non-jointly utilised machines (Salvadori 1999). At first, it turns out that the Uniform Efficiency Path Axiom may not be able to apply directly to the case with jointly utilised machines. However, it can be shown that the properties of the non-transferable jointly utilised machines model can be obtained if the Uniform Efficiency Path Axiom with some variations holds. More specifically, if a modified Uniform Efficiency Path Axiom holds, then the determination of the cost-minimising technique is independent of the structure of consumption, a property of the model with non-transferable jointly utilised machines as shown by Salvadori (1988). It can also be shown that if the rate of growth equals the rate of profit, then the the prices in terms of wage rate of finished goods are uniquely determined even if there exists more than one cost-minimising technique. Further, the modified Uniform Efficiency Path Axiom is not only a sufficient condition, but also a necessary condition for the determination of the cost-minimising technique being independent of the structure of consumption.

Chapter 3 is a note which makes a simple generalisation of the singular rent model built by Salvadori (1983). In much of the literature of modern reformulation of classical rent theory, requirements for use are explicitly or implicitly assumed to be exogenous given. The only exception is Salvadori (1983)'s model, in which he discovered a new variety of rent, called singular rent, when requirements are functions of income distribution. In order to separate singular rent from other kinds of rent, Salvadori only considered a single system of production. With regard to land, there always exists a problem of choice of technique. What problems may occur to the singular rent model if there exist many techniques that can be chosen? How to deal with the problem of choice of technique in this model? These are the questions that Chapter 3 seeks to investigate. This chapter makes a simple generalisation of the singular rent model by introducing many agricultural processes. Hence the problem of choice of technique arises. The cost of this generalisation is mixing singular rent with intensive rent. The main results of this chapter can be summarised as follows. With respect to singular rent alone, it is shown that there may exist more than one cost-minimising technique even though joint production is set aside, and there may be no cost-minimising technique even though all techniques are feasible. However, in the latter case, the system determining intensive rent is feasible and cost-minimising.

Chapter 4 is devoted to the problem of exhaustible resources. This chapter investigates the problem of exhaustible resources using a dynamic input-output model with classical features. The studies of exhaustible resources in the classical framework are in a state of intense debate, which concerns whether or not the classical economics is capable to deal with the problem of exhaustible resources, and how to determine royalties of exhaustible resources. In this literature, Kurz and Salvadori's contributions, from which Chapter 4 is inspired, are most comprehensive and consistent. In their later contributions (Kurz and Salvadori 2009, 2011), Kurz and Salvadori made it clear that the differences between the analysis by the classical economists, especially by Ricardo and that by Hotelling are due to the different assumptions. The classical analysis on exhaustible resources is not barren nor inferior to the analysis of Hotelling, but is complementary to the latter. Both ideas can be incorporated into a single framework. Following the suggestions and hints given by Kurz and Salvadori (1995, 2009, 2011), this chapter seeks to make a further reconciliation between the analyses of Ricardo and Hotelling on exhaustible resources, by introducing resource searching activities. The model is based on a given real wage rate and a given consumption vector, and it is shown that the paths of prices, royalties, rents, intensities of production and searching processes can be determined once a sequence of profit rates and the initial amounts of commodities and resources are given. This chapter also discusses circumstances under which commodity prices are constant when exhaustible resources exist, based on the model presented. It further confirms that Ricardo's analysis on exhaustible resources is correct if some conditions are satisfied, and that both his analysis and Hotelling's are helpful in improving our understandings in the issue of exhaustible resources.

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Chapter 1

Fixed Capital Models in the Sraffa Framework: A Survey

1.1 Introduction

It is well known that both von Neumann (1945)[1] and Sraffa (1960) revived the classical idea of treating fixed capital as a special case of joint production. Von Neumann expressed clearly that "wear and tear of capital goods are to be described by introducing different stages of wear as different goods" (von Neumann, 1945, p. 2), and although he did not mention any classical economist, scrutiny of the assumptions and methods which he used, such as the asymmetrical treatment of income distribution, the rules of free goods and the method of choice of technique, shows that his analysis more convincingly belongs to the classical tradition of economic thought rather than the neo-classical tradition (Kurz and Salvadori, 1993). Sraffa also rediscovered the classical treatment of fixed capital as joint production independently,[2] and in his opinion, fixed capital is the "leading species" of joint production, an idea which can be dated back to the classical economists, Torrens, Ricardo,

[1]The original paper by von Neumann was published in German in 1937 and translated into English in 1945. In this chapter we refer to the English version.

[2]For Sraffa's consecutive attempts in solving the problem of fixed capital in a objective way, see Kurz and Salvadori (2005).

Malthus and Marx.[3]

The joint production treatment of fixed capital is not only different from, but is also more logically consistent than the neo-classical treatment. In the neo-classical theory, fixed capital is usually treated as a stock, a"factor of production", which is either a given quantity of a single magnitude in the traditional version, or a given vector of capital goods in the modern or neo-Walrasian version. Either way, the neo-classical capital theory suffers from insurmountable difficulties that cannot be solved within its framework (Petri, 2004). Even if we ignore the logical inconsistencies and the critiques raised by Sraffa's work and subsequent contributions in capital controversy to the neo-classical capital theory, it is still unsatisfactory to treat fixed capital as a stock since fixed capital has both characteristics of "flow" and "stock" (Pasinetti, 1980). However, such intricacies arising from the "stock" and "flow" properties of fixed capital can be easily avoided by the method of joint production, i.e., the classical treatment, because the machines of the same type but different ages are treated as different commodities. A machine utilised in production is treated as an input, and the same machine after the production period is treated as a different commodity produced jointly with the finished good. As a consequence, all machines become flows, and thus

[3] There exists a debate about whether the method of treating fixed capital as joint product was adopted by the classical economists and Marx. Moseley (2009) argues that Sraffa's attribution of the joint product method of treating fixed capital to Marx is misleading at best and totally wrong at worst, as is the same attribution to other classical economists like Torrens and Malthus. Gehrke (2011), in re-examining the "Reference to the literature" in Sraffa's book, argues that Moseley's view is difficult to sustain, and that it is absurd to think that the classical economists adopted the joint product method to treat fixed capital only if they used the method in exactly the same way as Sraffa, since the classical economists did not have the same analytical tools at their disposal. Moseley (2011) rebutted that his earlier conclusion still held, and further argued that Ricardo did not adopt the joint product method, and that Marx's "transformation of value" method was superior to the joint product method. Moseley's contention that Marx's theory was superior was based on his critique of the "flaw" (used machines are not sold on the market and as a consequence the rate of profit across machines of different ages in the same industry cannot be equal) and "unrealistic assumptions" of the latter. This argument is hard to sustain for the following reasons. First, even if there is no market for selling old machines, the prices of such machines determined in the Sraffa system are book values, which can give the correct depreciation and annual charge of the machines. Second, those "unrealistic assumptions" (such as there is only one single fixed capital good in each industry) are not needed in some fixed capital models.

the intricacies disappear. Moreover, the classical treatment of fixed capital can analyse certain economic problems that the "fund-flow" approach, proposed by Georgescu-Roegen, cannot deal with satisfactorily, irrespective of the fact that the latter may appear to have the same structure as the former (Kurz and Salvadori, 2003). For instance, the "fund-flow" approach cannot satisfactorily provide answers to questions like why some machines become obsolete prematurely, or why some equipment is left idle rather than operated with the full degree of utilisation. Especially with regard to the problem of choice of technique, the "fund-flow" approach may yield misleading or incorrect answers in identifying the cost-minimising technique because this approach may unify several production processes within one entity and hide the fact that some of these processes are not cost-minimising (see the example given by Kurz and Salvadori, 2003). What is more important is that the classical treatment of fixed capital does not make illegitimate prior assumptions about which of the underlying techniques will be used, unlike the "fund-flow" approach. Therefore, the classical treatment is superior to the "fund-flow" approach. In addition, the treatment of fixed capital as joint production is more general than other multi-sectoral fixed capital models (Lager, 1997; Lager, 2006).

Sraffa presented a simple model with a single machine of constant efficiency. The model was widely generalised, and such generalisations can be roughly categorised by two criteria according to the properties of old machines: joint utilisation and transferability. The former criterion concerns whether or not an old machine can be jointly utilised with other old machines in the same production process, and the latter concerns whether or not an old machine is used in different sectors. However, as will be observed below, if machines always work with constant efficiencies as in the model presented by Sraffa, the above classification is unnecessary, and it only matters when machines have variable efficiencies. Hence, when machines have variable efficiencies, the fixed capital model can be a non-transferable non-jointly utilised machines model, or a non-transferable jointly utilised machines model, or a transferable non-jointly utilised machines model, or a transferable jointly utilised machines model. This chapter seeks to make a brief survey of the development of such fixed capital models in the Sraffa framework.

With regard to fixed capital, there always exists a problem of choice of technique, which concerns the patterns of utilisation of machines and the choice of the economic lifetime of machines (Kurz and Salvadori, 1995). In this chapter we only discuss the latter problem,[4] and we focus on the properties of the cost-minimising technique, which will be defined below.

The chapter is organised as follows. Before embarking on the main subject, we start with some definitions and a summary of some common assumptions used in fixed capital models. Specific assumptions corresponding to each model are given in the corresponding section. In section 1.3 we discuss the model with machines of constant efficiencies. Following section 1.3 we consider models with machines of variable efficiencies based on the above classification. Section 1.4 discusses the non-transferable, non-jointly utilised machines model, followed by section 1.5, which deals with the non-transferable, jointly utilised machines model. In section 1.6 we consider a model built by Salvadori (1999), a model with transferable, non-jointly utilised machines that have uniform efficiency paths, which will be explained below. Section 1.7 is devoted to a brief discussion on the issue of transferable machines from a different perspective proposed by Bidard (2016). The next section discusses a model with transferable and jointly utilised machines under the condition of a variation in the uniform efficiency paths. The last section concludes.

1.2 Common Assumptions and Basic Definitions

In order to distinguish the models of fixed capital from those of pure joint production, some assumptions are usually needed. In this section we first summarise some common assumptions used in fixed capital models, and leave the specific assumptions corresponding to each kind of fixed capital model in the corresponding section below. Proceeding in this way makes the

[4]For a systematic discussion on the patterns of utilisation of durable capital goods in the Sraffa framework, see Kurz (1986; 1990), Kurz and Salvadori (1995, Chap.7, Sect. 7.).

differences between these models easier to understand and the characteristics of each model become clearer.

These common assumptions usually include:[5] first, a division of all commodities into two groups, namely finished goods and old machines. The former can be used as both consumption goods and means of production, while the latter are never used as consumption goods, but may be used as means of production. It should be noted that new machines belong to finished goods. Second, each production process produces one and only one finished good, and may produce a quantity of old machines. In other words, there is no "pure" joint production: finished goods are not assumed to be jointly produced, and the only possible joint products are old machines. Therefore, it is possible to define a sector which comprises all the processes producing the same finished good. Third, old machines are assumed to be disposed of freely at any time with zero scrap value.

Next we give some basic definitions concerning production technology and the cost-minimising technique. The problem of choice of technique can be dealt with by two equivalent approaches, namely the indirect approach and the direct approach (Kurz and Salvadori, 1995). In the indirect approach, a single technique is first defined, and then a cost-minimising technique is chosen from the set of all techniques based on the cost-minimising criterion. By contrast, in the direct approach, a cost-minimising technique is chosen from the whole technology which comprises all production processes, and in this approach a cost-minimising technique can be determined without investigating non-cost-minimising techniques. In this chapter we use the direct approach since, compared with the indirect approach, the problem of joint production can be more easily handled with the former.

Assume that there are n commodities which can be produced by m constant-returns-to-scale processes, and each is represented by a triplet (a_i, l_i, b_i), $i = 1, 2, \cdots, m$, where a_i is a semi-positive commodity input vector,[6] l_i is a non-negative labour input scalar, and b_i is a semi-positive commodity out-

[5] For a more formal representation of these assumptions, readers can refer to Kurz and Salvadori (1995, Chap. 7 and Chap. 9)

[6] All vectors are column vectors, and the transpose of a vector or a matrix is denoted by a superscript T.

put vector. Reorder the commodities such that the first s commodities are finished goods, and the others are old machines. The whole technology is represented by the following matrices:

$$A = [a_1, a_2, \cdots a_m]^T$$

$$l = [l_1, l_2, \cdots l_m]^T$$

$$B = [b_1, b_2, \cdots b_m]^T$$

With regard to production technology, the following assumptions hold:

Assumption 1.1. *It is impossible to produce commodities without any material inputs, or:*

$$e_i^T A \geq 0, \qquad i = 1, 2, \cdots, m$$

Assumption 1.2. *All the commodities are producible, or:*

$$Be_j \geq 0, \qquad j = 1, 2, \cdots, n$$

Assumption 1.3. *Labour enters directly or indirectly into the production of all commodities, or:*

$$\forall \varepsilon > 0, (x \geq 0, x^T(B - \varepsilon A) \geqq 0^T) \Rightarrow x^T l > 0$$

Let p be the price vector, and let w be the wage rate. The rate of profit r is assumed to be exogenous. As regards the quantity system, let x be the intensity vector, and let d be the requirements for use. More specifically, d is assumed to have the following form:

$$d = gx^T A + c \tag{1.1}$$

in other words, the economy is assumed to undergo steady growth at the rate g, which satisfies $g \leqq r$, and c is the consumption vector. Since old machines are not assumed to be consumed, some of the first s elements of c are positive and the others are zeros, or $c^T = (c_s^T, 0^T)$, where c_s is an s-dimensional semi-positive vector. The numéraire is defined by a vector f, which is any given semi-positive vector whose positive elements refer to commodities that are certainly produced. In the condition of free competition, the following system holds:

$$[B - (1+r)A]p \leqq wl \tag{1.2a}$$
$$x^T[B - (1+r)A]p = wx^T l \tag{1.2b}$$
$$x^T[B - (1+g)A] \geqq c^T \tag{1.2c}$$
$$x^T[B - (1+g)A]p = c^T p \tag{1.2d}$$
$$f^T p = 1 \tag{1.2e}$$
$$p \geqq 0, x \geqq 0, w \geqq 0 \tag{1.2f}$$

Inequality (1.2a) means that no production process can obtain extra profits, and equation (1.2b) means that if there is one process which incurs extra costs, then the corresponding intensity of this process is zero. Inequality (1.2c) means that the amounts of commodities produced cannot be less than the amounts of commodities required by the production plus requirements for use, and equation (1.2d) means that if there exists one commodity which is overproduced, then its price is zero. Equation (1.2e) is the numéraire equation, and the meaning of inequalities (1.2f) is obvious.

If there exists a solution (x^*, p^*, w^*) to the above system, then there is said to exist a cost-minimising technique, and the following condition guarantees the existence of the solution (Kurz and Salvadori, 1995, Chap. 8).

There exists a non-negative vector z such that the following inequality holds:

$$z^T[B - (1+r)A] \geqq c \tag{1.3}$$

The price vector p^* (the wage rate w^* and intensity vector x^*) is called the long-period price vector (wage rate and intensity vector). A cost-minimising technique (A^*, l^*, B^*) is constituted by the processes which do not incur extra costs at the long-period prices p^* and wage rate w^* and which can produce the requirements for use with a positive intensity vector. More formally, we have the following system:

$$[B^* - (1+r)A^*]p^* = w^* l^* \tag{1.4a}$$

$$\bar{x}^{*T}[B^* - (1+g)A^*] = c^T \tag{1.4b}$$

where $\bar{x}^*$ is obtained from x^* by deleting the zero elements. In the following sections we will summarise the properties of the cost-minimising technique of each model. It is well known that, with regard to the problem of choice of technique, joint production causes many complexities: requirements for use, or "demand", may have an influence in determining the cost-minimising technique, the cost-minimising technique may not exist even if all techniques are feasible, the uniqueness of commodity prices may not hold if there is more than one cost-minimising technique, and so forth (see for instance, Schefold, 1989; Salvadori, 1982; Salvadori, 1985; Kurz and Salvadori, 1995; or the survey by Salvadori and Steedman, 1988). However, for fixed capital models, some of these complexities do not exist.

1.3 Model with Machines of Constant Efficiencies

We first discuss the model presented by Sraffa (1960), a model with a single machine which has constant efficiency. This arrangement is not only due to the fact that it is a seminal contribution in this literature, but also because of the fact that if machines always work with constant efficiencies, then the classification based on transferability and joint utilisation becomes unnecessary, as will be shown below.

We exclude perennial machines for convenience, which means that for each machine there exists a maximum physical life, after which the machine will be disposed of with zero scrap value.

By constant efficiency, Sraffa means that "[t]he quantities of means of production, of labour and of the main product [that is the finished good] are

Table 1.1: An Example: Machine of Constant Efficiency

	Inputs								Outputs					
	$S-1$	M_0	M_1	$\cdots$	M_{t-1}	$\cdots$	L		$S-1$	M_0	M_1	$\cdots$	M_{t-1}	$\cdots$
(1)	$a^T_{(s-1)}$	m_0	0	$\cdots$	0	$\cdots$	l_1	$\rightarrow$	$b^T_{(s-1)}$	0	m_1	$\cdots$	0	$\cdots$
(2)	$a^T_{(s-1)}$	0	m_1	$\cdots$	0	$\cdots$	l_1	$\rightarrow$	$b^T_{(s-1)}$	0	0	$\cdots$	0	$\cdots$
						$\vdots$	$\vdots$							
(t)	$a^T_{(s-1)}$	0	0	$\cdots$	m_{t-1}	$\cdots$	l_1	$\rightarrow$	$b^T_{(s-1)}$	0	0	$\cdots$	0	$\cdots$

equal in the several processes in accordance with the assumption of constant efficiency during the life of the machine" (Sraffa, 1960, p. 65). This can be illustrated as follows. Assume that a machine M, for instance, whose physical life is t years, is used in the production of finished good 1. Then there exist t processes producing finished good 1 using machine M of different ages as means of production. Let the sth commodity represent the new machine M_0, and let the $(s+1)$th commodity represent the one-year-old machine M_1, and so on. Let the $(s+t-1)$th commodity be the $(t-1)$-year-old machine M_{t-1}. If machine M always works with constant efficiency, then for all processes producing finished good 1 using machine M, the first $s-1$ elements of commodity input and output vectors and the labour input scalar are the same. All these processes producing finished good 1 using machine M are listed in Table 1.1, where $a_{(s-1)}$ and $b_{(s-1)}$ represent the first $s-1$ elements of the commodity input vector and output vector, respectively.

If machine M always works with constant efficiency, then the problem of choice of technique only concerns whether or not it is profitable to use machine M in the production of finished good 1 at the given rate of profit. Assume that the processes listed in Table 1.1 are cost-minimising at the given rate of profit r. Then we have the following subsystem producing finished good 1, a system similar to Sraffa's (Sraffa, 1960, p. 65):

$$(a^T_{(s-1)}p^*_{(s-1)} + m_0 p^*_{m_0})(1+r) + l_1 w^* = b^T_{(s-1)}p^*_{(s-1)} + m_1 p^*_{m_1} \tag{1.5a}$$
$$(a^T_{(s-1)}p^*_{(s-1)} + m_1 p^*_{m_1})(1+r) + l_1 w^* = b^T_{(s-1)}p^*_{(s-1)} + m_2 p^*_{m_2} \tag{1.5b}$$

$$\cdots \tag{1.5c}$$

$$(a^T_{(s-1)}p^*_{(s-1)} + m_{t-1}p^*_{m_{t-1}})(1+r) + l_1 w^* = b^T_{(s-1)}p^*_{(s-1)} \tag{1.5d}$$

where $p^*_{(s-1)}$ represents the first $s-1$ elements of p^*. Following Sraffa, if we multiply the above equations respectively by $(1+r)^{t-1}$, $(1+r)^{t-2}$, $\cdots$, $(1+r)$, 1, and add them, the old machines of different ages are cancelled out, we then get an integrated process producing finished good 1 without using old machines. Consequently for the cost-minimising technique, some properties similar to those of single production can be obtained easily, for instance, determination of the cost-minimising technique is independent of the structure of requirements for use, the prices in terms of the wage rate of finished goods are uniquely determined, and etc.

Although the model Sraffa presented is simple and may be special, it emerges that many assumptions made in some of the following models such as non-joint utilisation or non-transferability of old machines are not necessary to obtain those good properties when machines have constant efficiencies.

If machines always work with constant efficiencies, then joint utilisation of old machines does not cause any complexities, as is shown by Roncaglia (1978). It is still possible to integrate the processes producing the same finished good using machines of different ages and types into one similar to a single production process. Hence the good properties similar to those of single production follow. Further, if the machines always work with constant efficiencies, then there is no need to assume that the old machines are non-transferable, as do many generalisations to Sraffa's model. We will discuss the models with transferable machines in detail in the later sections. Given the above considerations, the model with machines of constant efficiencies is not as special as it appears.

In what follows, we will discuss the models with machines of variable efficiencies based on the two criteria mentioned above: transferability and joint utilisation.

1.4 Model without Transferable or Jointly Utilised Machines

Sraffa's model was first generalised into a model with non-transferable and non-jointly utilised machines that have variable efficiencies. This kind of fixed capital model, simply called a single machine model in this survey, has been widely investigated (Baldone, 1980; Schefold, 1980; Varri, 1980; Kurz and Salvadori, 1994; Kurz and Salvadori, 1995, chap. 7). One result of this generalisation is that the optimal life of a machine may not be the same as its physical life. If the efficiency of a machine is constant, there is no reason to stop using the machine till the end of its physical life. By contrast, if the efficiency of a machine is decreasing, then this machine may become economically obsolete before its physical life comes to an end. Actually, the optimal life for using the machine can be endogenously determined by the cost-minimising technique and is not independent of income distribution. In the following passage in this section, we will introduce this model and discuss its properties.

Except for those mentioned in section 1.2, some specific assumptions are required to isolate this model from other fixed capital models. First, the transferability of old machines is ruled out. That is to say, for each old machine k which is produced by a process producing, for instance, finished good j, any process producing another finished good other than j does not use old machine k as an input, nor does it produce k as an output. Second, old machines are not jointly utilised. This assumption means that each process uses no more than one old machine, and each process produces no more than one old machine. Old machines produced by the processes that use finished goods alone are called one-year-old machines, and those machines produced by the processes that use finished goods and one-year-old machines are called two-year-old machines, and so on.

If we use t_1 to denote the number of old machines that are used in sector 1 (for instance, assume that two types of machines M and N, which last for $\tau_1 + 1$ and $\tau_2 + 1$ years, are used in the production of finished good 1, then $t_1 = \tau_1 + \tau_2 - 2$), and so on, and assume that there exist m_i processes that can produce commodity i, then after a proper reorder, the matrices A and B have the following forms:

$$A = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} A_{11} & A_{1t_1} & & & \\ \hline A_{21} & & A_{2t_2} & & \\ \hline \vdots & & & \ddots & \\ \hline A_{s1} & & & & A_{st_u} \end{array} \right) \end{array} \tag{1.6}$$

$$B = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} B_{11} & B_{1t_1} & & & \\ \hline B_{21} & & B_{2t_2} & & \\ \hline \vdots & & & \ddots & \\ \hline B_{s1} & & & & B_{st_u} \end{array} \right) \end{array} \tag{1.7}$$

where A_{i1} and B_{i1} represent the finished good inputs and outputs of sector i, respectively. In addition, only the ith column of B_{i1} is positive; other columns are nought. A_{it_i} (B_{it_i}) represents the old machines used in (produced by) sector i (if sector i does not use old machines, then the columns containing $A_{it_i}(B_{it_i})$ are deleted). Note that there may exist more than one type of old machine which can be used in the production of commodity i, yet not jointly utilised in a single process. Hence in each row of A_{it_i} (B_{it_i}), there exists at most one positive element, and the others are zeros.

Let (A^*, l^*, B^*) be a cost-minimising technique. For a single machine model, it can be proved that (A^*, l^*, B^*) has the following properties that are similar to those of single production (Baldone, 1980; Schefold, 1980; Varri, 1980; Kurz and Salvadori, 1994; Kurz and Salvadori, 1995, chap. 7): first, determination of the cost-minimising technique is independent of the structure of requirements for use, provided that old machines do not enter into the consumption vector. Second, the prices in terms of the wage rate of actually produced finished goods are uniquely determined even if there exists more than one cost-minimising technique. Third, the prices in terms of the wage rate of finished goods are increasing functions of r.

These properties follow from the fact that there exists a vector $x_i(g)$ for each sector i using old machines such that $x_i^T(g)B_{i1}^* = e_i^T$ and $x_i^T(g)[B_{it_i}^* - (1+g)A_{it_i}^*] = 0$, where e_i is the ith unit vector. $(x_i^T(g)A_{i1}^*, x_i^T(g)l_{(i)}^*, e_i)$ is called a "core-process" (Kurz and Salvadori, 1995) or an integrated process producing finished good i, where $l_{(i)}^*$ is a vector constituted by the labour input scalars used in the processes producing finished good i and using old machines. Since the matrix formed by all these core processes has the same characteristics as the single production technique, the above properties can be obtained easily. After the prices of finished goods are obtained, the prices of old machines used in the cost-minimising technique can be determined sequentially.

Up to now we have not discussed the problems of depreciation and efficiency in this model. The depreciation of an old machine for one production period is nothing but the change in price of that machine over that period. Assume that one type of machine lasts for $(t+1)$ years in the cost-minimising technique, and let $p_0(r)$, $p_1(r)$, $\cdots$, $p_t(r)$ be the prices of this machine type of ages 0, 1, $\cdots$, t, respectively. The depreciation of the i-year-old machine is (Kurz and Salvadori, 1995, Chap. 7):

$$M_i(r) = p_i(r) - p_{i+1}(r) \quad (i = 0, 1, \cdots, t-1) \tag{1.8a}$$

$$M_t(r) = p_t(r) \tag{1.8b}$$

The annual charge relative to the i-year-old machine is:

$$Y_i(r) = (1+r)p_i(r) - p_{i+1}(r) \quad (i = 0, 1, \cdots, t-1) \tag{1.9a}$$

$$Y_t(r) = (1+r)p_t(r) \tag{1.9b}$$

which is the depreciation plus the profit earned by the capital in the form of the machine in the corresponding year. The efficiency of an i-year-old machine is defined as constant, increasing or decreasing if the annual charge of that machine is equal to, lower than, or higher than the annual charge of the $(i+1)$-year-old machine of the same type. Since the prices of machines are dependent on the condition of income distribution, depreciation, annual charge and efficiencies of machines are in general not independent of the condition of income distribution.

It should be noted that the above definition of constant efficiency is not in contradiction with Sraffa's definition. It can be checked that in system (1.5), the annual charges of machine M of different ages are always the same. Sraffa also wrote: " Supposing a machine 'm' to work with constant efficiency throughout its life, the annual charge to be paid for interest and depreciation in respect of it must be *constant*, if the price of all units of the product is to be uniform" (Sraffa, 1960, section 75, p. 64, emphasis added.).

Indeed it is possible to prove (see appendix) that in any (cost-minimising) technique if the annual charge relative to i-year-old machine is equal to the annual charge relative to the j-year-old machine, any $j \neq i$, at each rate of profit, then Sraffa's definition of constant efficiency holds.

In what follows we will adopt the definition of efficiency using the annual charge. As will be observed below, efficiency is very important when dealing with transferable machines. We leave that to section 1.6.

1.5 Model with Non-transferable and Jointly Utilised Machines

As stated in section 1.3, Roncaglia (1978) presented a model with non-transferable, jointly utilised machines which have constant efficiencies.[7] It was Salvadori (1988a; 1988b) who built a general model with non-transferable, jointly utilised machines.

Since the non-transferability assumption still holds, the technological ma-

[7]Roncaglia (1978) provided two generalisations to Sraffa's model: one is a model with non-transferable and non-jointly utilised machines with variable efficiency, and the other is a model with non-transferable, jointly utilised machines with constant efficiency. Although non-transferability is not explicitly assumed by Roncaglia (1978), he seems, like Sraffa, to suggest that if machines of the same type and the same age are used in the production of different finished goods, they should be treated as different machines which have different prices: "It is enough to think of the prices of machines of a given age and type -in general- as being different (because their circumstances of use have been different) in relation to the production process in which they have been used $\cdots$" (Roncaglia, 1978, p. 46.). Hence, the non-transferability is implicitly assumed. However, if the efficiencies of machines are always constant, the assumption of non-transferability is unnecessary.

trices A and B still have the forms like equations (1.6) and (1.7). However, the non-joint utilisation assumption is ruled out. Therefore, in each row of A and B, there can exist more than one element that is positive.

We still focus on the properties of the cost-minimising technique. Salvadori (1988a; see also Kurz and Salvadori, 1995, Chap. 9.) proved that determination of the cost-minimising technique is independent of the structure of consumption, provided that old machines are not consumed, although the investment (rate of growth) may have an influence on the determination of the cost-minimising technique. The latter result follows from the fact that an old machine may be overproduced even if it is utilised in the production given that old machines can be jointly utilised, and as a consequence, the change of relative intensities of the actually operated processes producing the same finished goods matters in determining whether or not an old machine is overproduced. The rate of growth can affect the determination of the relative intensities of actually operated processes of finished goods, hence mattering in the determination of prices.

The former result can be represented as follows. Assume that x^*, p^* and w^* are a solution to system (1.2) for a given consumption vector c_1. Then for another consumption vector $c_2 \neq c_1$, there exists another solution x^{**}, p^* and w^* to the same system. The proof of this result can be shown briefly as follows. Let x^* be partitioned as follows: $x^* = [x_1^{*T}, x_2^{*T}, \cdots, x_s^{*T}]^T$, where x_i^* is the intensities corresponding to sector i. Define a square matrix Q, whose ith row is $x_i^{*T} B_{1i}$, and a square matrix H, whose ith row is $x_i^{*T} A_{1i}$. Inequality (1.2c) can be represented as follows:

$$e^T[Q - (1+g)H] \geqq c_{1s} \tag{1.10a}$$

$$x_i^{*T}[B_{it_i} - (1+g)A_{it_i}] \geqq 0 \quad i = 1, \cdots, u \tag{1.10b}$$

where c_{1s} is the vector composed by the first s elements of c_1.

From inequality (1.10a) we know that matrix $[Q - (1+g)H]$ is invertible and the inverse is semi-positive (Kurz and Salvadori, 1995, Mathematical Appendix, Theorem A. 3. 1). Hence there exists a vector v such that $v^T[Q - (1+g)H] = c_{2s}$, where c_{2s} is a vector composed by the first s elements of

c_2. Further, a scalar multiplication of x_i will not change the inequalities (1.10b), i.e., $v_i x_i^*$ $(i = 1, \cdots, s)$ still satisfy (1.10b), where v_i is the ith element of v. The existence of vector v implies that there exist x^{**}, p^*, w^* as a solution to system (1.2) for a given consumption vector c_2, where $x^{**} = [v_1 x_1^{*T}, \cdots, v_s x_s^{*T}]^T$.

Salvadori (1988a) also proved that if the rate of growth equals the rate of profit ($g = r$), then the prices in terms of the wage rate of finished goods that are actually produced are uniquely determined even if there is more than one cost-minimising technique. This result may not hold if $g \neq r$. In addition, the uniqueness of prices in terms of the wage rate of old machines may not hold even if $g = r$.[8]

From the above analysis we can see that there is no need to introduce "ages" and "types" of machines in order to determine the cost-minimising technique. These two definitions are required when dealing with problems of depreciation and efficiency. Salvadori (1988a; see also Kurz and Salvadori, 1995, Chap. 9) gave an assumption that defines the 'ages" and "types" of machines. The analyses of depreciation and efficiency are the same as those in section 1.4, and will not be restated here. There exists only one point that needs emphasis: since old machines can be jointly utilised, the depreciation and efficiency of one particular machine is not only determined by the circulating capital goods, labour input utilised with it and the output produced with it, but is also influenced by the other machines jointly utilised with it. In other words, the efficiencies of jointly utilised machines are interdependent (Roncaglia, 1978; Lager, 1997) because the prices of old machines are interdependent.

[8]This can be briefly explained as follows: assume that there exist two processes involving the same commodities (including old machines) in two cost-minimising techniques (say technique 1 and 2) at the given rate of profit. Further assume that two types of old machines M and N are produced by these two processes. The quantities of M and N produced in technique i is m_i and n_i, respectively ($i = 1, 2$). The annual charges of machines M and N are $C_M^{(i)}$ and $C_N^{(i)}$, where the superscript i represents technique i, $i = 1, 2$. The uniqueness of prices in terms of the wage rate of finished goods only implies that for these two processes, $(C_M^{(i)} m_i + C_N^{(i)} n_i)/w_i$ are equal in these two processes, where w_i is the wage rate of technique i. However, since old machines can be jointly utilised in the same process, the above equality does not guarantee that the prices of the same old machines are the same in these two processes.

Salvadori (1988a) also made a variant to the above model by replacing the assumption of free disposal with an assumption that the entire amount of scrapped machines is fully utilised, directly or indirectly, in the production of the finished goods that they are produced with, and he showed that the above results still hold.

1.6 Model with Transferable and Non-jointly Utilised Machines

In the above two sections, we summarised some properties of the cost-minimising technique of the models without transferable machines. Non-transferability is crucial for obtaining these properties. In general, the existence of transferable machines will cause some complexities of pure joint production, because the system is “interlocked” (Schefold, 1989). Under some well-defined circumstances, however, the transferability of old machines will not cause any complexities, as is shown in this section.

The difficulties caused by transferable machines are suggested by Sraffa: if a machine is used in different industries (sectors), then this machine may have different working lives, and its efficiency may be different even if its working lives are the same. However, “ [i]f in all the industries the machine had the *same working life and constant efficiency*, the book-values for each age would be equal in all of them, since the annual charges would all be equal to the annuity described in § 75” (Sraffa, 1960, section 78, p. 67, emphasis added.).

Hence, it seems that if the transferable machines used in different sectors have the same working lives and constant efficiencies, then transferability will cause no problems. It is immediately understood that the above sentences by Sraffa are correct. Hence the problem is close at hand: what happens if the efficiencies of machines are not constant, but are independent of the sectors?

The suggestion made by Sraffa was developed by Salvadori (1999). He proposed a model with non-jointly utilised but transferable machines whose

Table 1.2: An Example with One Transferable Machine

	Inputs						Outputs			
	Maize	Wheat	M_0	M_1	Labour		Maize	Wheat	M_0	M_1
(1)	1/15	1/5	1	0	1/2	→	1	0	0	1
(2)	3/20	1/10	0	1	1/2	→	1	0	0	0
(3)	4/15	2/5	1	0	1	→	0	1	0	1

efficiencies are not constant but are still independent of the sectors in which they are used. Except for the assumptions mentioned in section 1.2 and the non-joint utilisation assumption in section 1.4, Salvadori dropped the assumption of non-transferability and made another assumption called the *Uniform Efficiency Path Axiom*, and he showed that the properties of the non-transferable, non-jointly utilised machines model still hold.

Let us first quote the Uniform Efficiency Path Axiom in full context and then give some explanations.

Uniform Efficiency Path Axiom (Salvadori 1999). *If a type of machine is used in the production of finished goods i and j $(i \neq j)$, then there is a vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ such that for each process (a_s^T, b_s^T, l_s) producing finished good i using a machine of that type (old or new) there is a process (a_t^T, b_t^T, l_t) producing finished good j such that the vector (a_t^T, b_t^T, l_t) is a linear combination of vectors (a_s^T, b_s^T, l_s) and $(a_{ij}^T, b_{ij}^T, l_{ij})$.*

We will use a simple example to give some explanations to the Uniform Efficiency Path Axiom.

Assume that there exists one transferable machine: a tractor (M), which lasts for 2 years, is used in the production of maize and wheat (two different finished goods). Further, assume that the processes listed in Table 1.2 are in the cost-minimising technique. M_0 represents a new tractor, which is a finished good, and M_1 is a one-year-old tractor (an old machine). Assume that land is not scarce and that the rate of profit is 1/4.

Since M_0 is a finished good, whose price is determined by other processes that are not listed here, we assume that its price is determined as 1/3. Taking maize as the numéraire, the above processes are able to determine the prices of wheat (p_2), M_1 (p_{M_1}), and the wage rate (w). It can be checked that $p_2 = 2$, $p_{M_1} = 1/4$ and $w = 1/2$.

If the Uniform Efficiency Path Axiom holds, then there exists a vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ and another process (process (4) which is represented by (a_4, b_4, l_4)) producing wheat such that the following equations hold:

$$(4/15, 2/5, 1, 0, 0, 1, 0, 1, 1) = \lambda_1(1/15, 1/5, 1, 0, 1, 0, 0, 1, 1/2) + \lambda_2(a_{ij}^T, b_{ij}^T, l_{ij}) \tag{1.11}$$

$$(a_4^T, b_4^T, l_4) = \lambda_3(3/20, 1/10, 0, 1, 1, 0, 0, 0, 1/2) + \lambda_4(a_{ij}^T, b_{ij}^T, l_{ij}) \tag{1.12}$$

The vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ is not normalised. For the sake of simplicity, let $\lambda_1 = 1$, and normalise process (4) such that it produces 1 unit of wheat. Then we have:

$$(a_{ij}^T, b_{ij}^T, l_{ij}) = (1/5, 1/5, 0, 0, -1, 1, 0, 0, 1/2)$$

Process (4) is listed in Table 1.3, and it can be shown that this process also belongs to the cost-minimising technique. We can explain the existences of the vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ and process (4) as follows. For processes (1) and (3), when the tractor is transferred from the sector producing maize to the sector producing wheat, a change in output (b_{ij}^T) requires a change in inputs (a_{ij}^T, l_{ij}). If the efficiency of the machine M is independent of the sectors, then we change the same input to process (2), and we can also produce wheat (process (4)).

From the analysis of the non-transferable, non-jointly utilised machines model we know that there exists a vector $x(g)$ such that a multiplication of $x^T(g)$ to processes (1) and (2) will yield a core process producing maize. Using the same $x^T(g)$ to multiply processes (3) and (4), we can also get a core process producing wheat. Hence the properties of the non-transferable,

Table 1.3: An Example with One Transferable Machine-Continued

	Inputs						Outputs			
	Maize	Wheat	M_0	M_1	Labour		Maize	Wheat	M_0	M_1
(4)	7/20	3/10	1	0	1	$\rightarrow$	0	1	0	0

non-jointly utilised machines model follow.

The Uniform Efficiency Path Axiom guarantees that if we treat the transferable machine M as two different machines (for instance tractor 1 and tractor 2), then these two machines have the same prices, hence the same path of depreciation and efficiency. In other words, the efficiency of the transferable machine is independent of the sector. Transferability causes no trouble if the Uniform Efficiency Path Axiom holds.

1.7 Bidard on Transferable Machines

In a recent contribution, Bidard (2016) discussed the models with transferable machines from a different perspective, pointing out that any model with transferable machines which has neat properties (the same as those of single production) can be associated to a model with non-transferable machines in which the prices of machines of the same ages but different types that are used in the production of different final goods, are the same at each rate of profit. He therefore argues that what matters is that the processes producing different finished goods but using the machines of the same type have equiprofitability. Hence he maintains that instead of the Uniform Efficiency Path it would be appropriate to say that there is equiprofitability. This, of course is not in contradiction with the argument developed by Salvadori (1999).

As stated in the previous section, Salvadori's argument generalises Sraffa's suggestion that transferable machines always have the same lives and constant efficiencies, into one such that the efficiencies of machines are not constant but are still independent of the sectors in which they are used. The assumption concerning efficiency independent of the sector is recognised as consisting in the fact that if a type of machine whose technical lifetime is n years and is used in the production of m final goods, then the linearly independent processes must be $(m + n - 1)$, whereas if efficiency depends on the sectors, then the linearly independent processes may be $(m \times n)$. Hence equiprofitability is not a better explanation than, but is a result of, the Uniform Efficiency Path Axiom.

The term equiprofitability can be applied to any kind of model (for instance, it may occur under certain circumstances in the case of single production or pure joint production), which may yield a mistaken impression that in the model with transferable machines, the property that processes using transferable machines producing different finished goods are equally profitable only holds by chance and consequently that the model has no economic relevance. However, under the circumstance that transferable machines have uniform efficiency paths, a circumstance which has its rationale and which is suggested by Sraffa, such equiprofitability is a definite, rather than an accidental result. What is more important is not to assert that if there is equiprofitability then the model preserves good properties, but to identify the conditions such that equiprofitability and these good properties hold.

Yet Bidard goes further, suggesting that there is no need to develop models with transferable machines with such a restrict assumption (the Uniform Efficiency Path Axiom) and that attention should be focused on applying the non-transferable machine model with only the remark that the model with transferable machines can be associated with one with non-transferable machines. However, even if this remark may be useful to understand the model and its properties, it does not undermine Salvadori's contribution. On the contrary, this remark further strengthens the idea that if the Uniform Efficiency Path Axiom holds, then the non-transferability assumption is unnecessary to obtain the neat properties that are close to those of single production.

1.8 Model with Transferable and Jointly Utilised Machines

Despite limiting his analysis to the case of transferable but non-jointly utilised machines, Salvadori conjectured that a similar formalism can be found for the case in which machines are used jointly, and this conjecture is correct. The Uniform Efficiency Path Axiom with some modifications can be applied to the case with transferable and jointly utilised machines, and it can be shown that under such an assumption the properties of the cost-minimising technique of the non-transferable jointly utilised machines model still hold (Huang, 2015).

To be more specific, in the model with transferable and jointly utilised machines, except for those assumptions listed in section 1.2, both the assumptions of non-transferable and non-jointly utilised machines are ruled out and are replaced by the Uniform Efficiency Path Axiom with some variations. The reasons why the Uniform Efficiency Path Axiom cannot be directly applied to the case with jointly utilised machines are as follows: first, the vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ mentioned in the Axiom may not exist. This is due to the fact that when old machines are not jointly utilised, the difference between two processes producing different finished goods and using an old machine of the same type and the same age only involves finished goods, while when old machines are jointly utilised, one transferable machine may be jointly utilised with old machines of different types and ages in the processes producing different finished goods. Hence the difference between the processes producing different finished goods and using the same transferable machine may involve old machines of different types and ages. Secondly, as pointed out in section 1.5, the efficiencies of machines are interdependent when they can be jointly utilised. Hence the efficiency path of one particular transferable machine may not be uniform due to the influences of other non-transferable machines jointly utilised with it in the same sectors.

In order to deal with jointly utilised machines, the Uniform Efficiency Path Axiom needs some modifications, and it is maintained in the following way: the efficiency path of a transferable machine is uniform *if* other machines jointly utilised with it are assumed to be non-existent (see Assumption 7 in Huang, 2015). This modified Uniform Efficiency Path Axiom has its strengths and weaknesses: on the one hand, it generalises the Uniform Efficiency Path Axiom because it allows jointly utilised machines (it is exactly the same as the Uniform Efficiency Path Axiom if joint utilisation of old machines is excluded). On the other hand, explanation of this assumption with the fact that transferable machines have uniform efficiency paths needs caution due to the fact that the efficiencies of machines are interdependent when they are jointly utilised. Only in one of the following situations can this assumption be explained as that transferable machines have uniform efficiency paths: (i) each transferable machine and other machines jointly utilised with it form a plant;[9] (ii) transferable machines of different ages are jointly utilised with other machines of the same ages and the same types; (iii) all other machines jointly utilised with transferable machines have constant efficiencies; (iv) the transferable machine itself has constant efficiency.

It can be shown that if the assumptions in section 1.2 and the modified Uniform Efficiency Path Axiom hold, then determination of the cost-minimising technique is independent of the structure of consumption, provided that the old machines are not consumed, and that the prices in terms of the wage rate of finished goods are positive. In addition, if the rate of growth equals the rate of profit ($g = r$), then the prices in terms of the wage rate of finished goods are uniquely determined even if there exists more than one cost-minimising technique. Further, the modified Uniform Efficiency Path Axiom is not only a sufficient condition, but also a necessary condition for the determination of the cost-minimising technique being independent of the structure of consumption.

1.9 Concluding Remarks

This chapter briefly surveyed the development of fixed capital models in the Sraffa framework. In order to separate fixed capital models from pure

[9]For the definition of a plant, see Kurz and Salvadori (1995, p. 207, p. 266). In this case, the efficiency path of the whole plant is uniform.

joint production, some assumptions are usually made. This chapter started with some common assumptions and proceeded to discuss all these models. Attention was focused on the properties of the cost-minimising technique. The models with machines of constant efficiencies were presented first, and it emerged that if machines always work with constant efficiencies, then it is not necessary to assume non-joint utilisation or non-transferability of old machines in order to obtain some neat properties. In what followed we discussed the generalisations of Sraffa's model, models with machines of variable efficiencies, according to the following two criteria: transferability and joint utilisation. The model with non-transferable and non-jointly utilised machines has good properties which are similar to those of single production, for instance, determination of the cost-minimising technique is independent of the structure of requirements for use. For the non-transferable, jointly utilised machines model, determination of the cost-minimising technique is independent of the structure of consumption, but may be influenced by investment (economic growth). Finally, models with transferable machines are discussed. Following Sraffa's suggestions on transferable machines, Salvadori (1999) showed that if the efficiencies of transferable machines are not constant but are still independent of the sectors in which they are used, then the model with transferable, but non-jointly utilised machines still has neat properties like those of single production. Transferability causes no complexities if the Uniform Efficiency Path Axiom holds. This assumption with some variations can also be applied to the case with transferable, jointly utilised machines, and it can be shown that the properties of the non-transferable, jointly utilised machines model are still obtainable.

Appendix

In this appendix, we will prove that for a cost-minimising technique, if a machine always works with constant efficiency irrespective of the rate of profit, then the definition of efficiency used in section 1.4 is equivalent to Sraffa's. Given that, if machines are jointly utilised, the efficiencies of machines of different types are interdependent, which further complicates this issue, we will prove the equivalence of these two definitions in a case with only non-jointly utilised machines.

Proposition 1.1. *Let* (A^*, l^*, B^*) *be a cost-minimising technique for any given* r *that belongs to* $[\underline{r}, \bar{r}]$. *If the annual charge relative to the* i*-year-old machine* M*, defined by system* (1.9)*, is always the same as the annual charge of the* $(i+1)$*-year-old machine of the same type for any rate of profit* r *which belongs to* $[\underline{r}, \bar{r}]$, $i = 0, 1, \cdots, t-1$, *where* t *is the maximum age of machine* M *produced by the processes producing finished good* j *in the cost-minimising technique, then for all processes producing finished good* j *using machine* M *as an input, the quantities of finished good inputs except the new machine and of labour input in order to produce* 1 *unit of finished good* j *are the same.*

Proof.

Let (a_i, l_i, b_i) and $(a_{i+1}, l_{i+1}, b_{i+1})$ represent two processes producing finished good j using machine M of age i and $i+1$ $(i = 0, 1, \cdots, t-1)$, respectively, and let the new machine M be the sth finished good. These two processes are normalised in such way that they produce one unit finished good j. The annual charge relative to the i-year-old machine is:

$$\begin{aligned} Y_{M_i}(r) &= (1+r)p^*_{M_i}(r) - p^*_{M_{i+1}}(r) \\ &= \frac{1}{m}[b^T_{i(s-1)}p^*_{(s-1)} - (1+r)a^T_{i(s-1)}p^*_{(s-1)} - l_i w^*] \end{aligned} \tag{1.13}$$

where m is the quantity of machine M in order to produce one unit of finished good j, $b_{i(s-1)}$, $a_{i(s-1)}$ and $p^*_{(s-1)}$ represent the first $s-1$ elements of b_i, a_i and p^*, respectively.

Since $b_{i(s-1)} = b_{i+1,(s-1)}$, if $Y_{M_i}(r) = Y_{M_{i+1}}(r)$, we have the following equation:

$$(1+r)a^T_{i(s-1)}p^*_{(s-1)} + l_i w^* = (1+r)a^T_{i+1,(s-1)}p^*_{(s-1)} + l_{i+1}w^* \tag{1.14}$$

or:

$$(1+r)[a^T_{i(s-1)} - a^T_{i+1,(s-1)}]\frac{p^*_{(s-1)}}{w^*} = l_{i+1} - l_i \tag{1.15}$$

From section 1.4 we know that the prices in terms of the wage rate are increasing functions of r, and an increase in r will increase the level of $(1+r)\frac{p^*_{(s-1)}}{w^*}$. Since the right hand of (1.15) is constant, we have $l_i = l_{i+1}$ and $a_{i(s-1)} = a_{i+1,(s-1)}$.

Q.E.D.

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Chapter 2

A Fixed Capital Model with Transferable and Jointly Utilised Machines in the Sraffa Framework

2.1 Introduction

Sraffa (1960) revised the classical idea of treating fixed capital as a special case of joint production, such that the machine entering into the production process is treated as a different commodity from the same machine after the production process. The generalisation of Sraffa's model by subsequent scholars can be roughly categorised according to two different criteria: joint utilisation and transferability. The former criterion concerns whether or not a machine is jointly utilised with other machines in the same process, while the latter concerns whether or not a machine is used in different sectors. Therefore, a fixed capital model can be a non-transferable, non-jointly utilised machines model (which will be called the non-transferable single machine model in this chapter), as in Sraffa (1960), Baldone (1980), Schefold (1980), Varri (1980), Kurz and Salvadori (1994, 1995 chap. 7), or a non-transferable jointly utilised machines model, following Roncaglia

(1978), Salvadori (1988a, 1988b), Kurz and Salvadori (1995 chap. 9), or a transferable non-jointly utilised machines model (which will be called the transferable single machine model), as in Salvadori (1999), or a transferable jointly utilised machines model.

Like single production, the non-transferable single machine model has good properties. For instance, the cost-minimising technique in the non-transferable single machine model is independent of demand. In the non-transferable jointly utilised machines model, economic growth may have an influence on choice of technique, but consumption does not. Conversely, fixed capital models with transferable machines are rarely studied. Most authors believe that introduction of transferability of machines will cause complexities of pure joint production. However, Salvadori (1999) shows that this worry is unnecessary when some conditions are satisfied. The idea of the non-transferable old machines put forward by Sraffa (1960, section 78, p. 66) seems imply that the troubles raised by the transferable old machines are due to the fact that the transferable machines have either different working lives in different sectors, or their efficiencies are non-constant in at least one sector, or both. Salvadori generalises Sraffa's suggestion by assuming that the efficiency of a transferable machine is not constant, but is still independent of the sector in which the machine is used (the assumption called Uniform Efficiency Path Axiom). Restricting his analysis within no joint utilisation of old machines (although he mentions that a similar formalism can be found for the case of jointly utilised machines), he shows that all the good properties of the non-transferable single machine model hold. This chapter will generalise Salvadori's model (1999) into a jointly utilised machines model, which means both joint utilisation and transferability are allowed. *The assumption of uniform efficiency paths of transferable machines still holds but with some modifications*. This chapter will show that the properties of the non-transferable jointly utilised machines model are still obtainable.

One anonymous referee suggests that the problem dealt in this chapter can be considered from a different view as follows. In an economy excluding the transferability of old machines, if the processes in two sectors using different old machines generated by the same new machine satisfy the Uniform

Efficiency Path axiom, then for the cost-minimising technique, the prices of these old machines used in these two sectors are the same, and these machines can be considered as transferable. This suggestion is interesting and is gratefully acknowledged. This chapter, however, introduces the transferability of old machines directly, and it shows that if the Uniform Efficiency Path axiom holds, the transferability of old machines causes no troubles compared to the non-transferability. Therefore the non-transferability assumption can be dropped if the Uniform Efficiency Path axiom holds.

The chapter is organised as follows: section 2.2 will give some basic definitions and assumptions used in later sections. The main theorems will be presented in section 2.3. They will prove that the cost-minimising technique is independent of consumption, and that prices in terms of the wage rate of finished goods are unique even if there exists more than one cost-minimising technique under the condition that the economic growth rate equals the profit rate. Section 2.4 will discuss the necessary condition for the choice of technique being independent of consumption. Finally, section 2.5 will provide some conclusions.

2.2 Basic Definitions and Assumptions

Assume that there are n perfectly divisible commodities produced by m perfectly divisible processes, and each process i $(i = 1, 2, \cdots, m)$ is represented by a triplet (a_i, b_i, l_i), where $a_i^T = (a_{i1}, a_{i2}, \cdots, a_{in})$ and $b_i^T = (b_{i1}, b_{i2}, \cdots, b_{in})$ are the non-negative input vector and output vector respectively, and l_i is the non-negative labour input scalar. The whole technology is represented by the triplet (A, B, l) which has the following form:

$$A = [a_1, a_2, \cdots, a_m]_{m\times n}^T$$

$$B = [b_1, b_2, \cdots, b_m]_{m\times n}^T$$

$$l = [l_1, l_2, \cdots, l_m]_{m\times 1}^T$$

The following assumptions hold:

Assumption 2.1. *It is impossible to produce commodities without any material inputs, or:*

$$e_i^T A \geq 0, \qquad i = 1, 2, \cdots, m$$

Assumption 2.2. *All the commodities are producible, or:*

$$Be_j \geq 0, \qquad j = 1, 2, \cdots, n$$

Assumption 2.3. *Labour enters directly or indirectly into the production of all commodities, or:*

$$\forall \varepsilon > 0, (x \geq 0, x^T(B - \varepsilon A) \geqq 0^T) \Rightarrow x^T l > 0$$

Now the assumptions of fixed capital model with both transferable and jointly utilised machines will be introduced.

Assumption 2.4. *Set N of the n commodities can be partitioned into three disjointed subsets S, K and H. $S \cap K = \varnothing$, $S \cap H = \varnothing$, $K \cap H = \varnothing$, $S \cup K \cup H = N$. S, K and H have the following properties:*

2.4.1 Commodities in $K \cup H$ are never used as consumption goods, i.e., the corresponding entries in the consumption vector c are zero.

2.4.2 Each process produces one and only one commodity in S, and may or may not produce an amount of commodities in $K \cup H$. Each process uses commodities in S as inputs, and may or may not use an amount of commodities in $K \cup H$.

2.4.3 If there is a commodity $k \in K$ produced by a process which produces a commodity $s_1 \in S$, then there is no other process which produces commodity $s_2 \in S$ ($s_2 \neq s_1$) such that it either uses k as an input or produces k as an output.

2.4.4 For each process producing commodity $j \in K \cup H$, there is another process with the same inputs and outputs except that commodity j is not produced.

In the following, use $\mathbb{T}$ as the union of K and H for short. In the above assumption, commodities in S will be called finished goods and commodities in $\mathbb{T}$ will be called old machines. Furthermore, commodities in K will be called *non-transferable* old machines and commodities in H will be called *transferable* old machines.

According to Assumption 2.4, a commodity can either be a finished good or an old machine, but never both. Assumption 2.4.1 implies that old machines are never used as consumption goods. Assumption 2.4.2 implies that there is no joint production of finished goods. Hence, it is possible to define a *sector*, which is constituted by the processes producing the same finished goods, i.e., all the processes producing finished goods i are called sector i ($i = 1, 2, \cdots, s$). Assumption 2.4.3 states that old machines in K cannot be transferred from one sector to another. Assumption 2.4.4 implies that old machines can be disposed of freely (with zero scarp value).

Suppose there are μ_i processes producing finished good i. Reorder all the processes in (A, B, l) in the following way: the first μ_1 processes produce finished good 1, then the following μ_2 processes produce finished good 2, and so on. Reorder the commodities in the following way: the first s commodities are in S, the following k commodities are in K, and the remaining commodities are in H. Let K_i and H_i be the sets of non-transferable old machines and transferable old machines used in sector i, respectively. Assumption 2.4 implies that $K_i \cap K_j = \varnothing$ for $i \neq j$, while for $i \neq j$, $H_i \cap H_j$ may or may not be empty. However, for each $H_i \neq \varnothing$, there exists at least one set H_h, $i \neq h$ such that $H_i \cap H_h \neq \varnothing$. Otherwise, $H_i \subset K_i$, which implies $H_i = \varnothing$.

Assumption 2.5. *Constant returns to scale prevails.*

Therefore, it is possible to normalise all processes by the finished goods they produce, that is if $j \in S$, $\sum_{k=0}^{j-1} \mu_k < i \leq \sum_{k=1}^{j} \mu_k$ ($\mu_0 \equiv 0$), then $b_{ij} = 1$.

In order to introduce efficiency and depreciation, we should distinguish different ages and types of machines.

Assumption 2.6. *It is possible to normalise the physical unit of commodities in $\mathbb{T}$ in the following way: there are two natural numbers u and v such that* *the set $\mathbb{T}$ can be partitioned both in u subsets $\mathbb{T}_{1*}$, $\mathbb{T}_{2*}$, $\cdots$, $\mathbb{T}_{u*}$, and in v subsets $\mathbb{T}_{*1}$, $\mathbb{T}_{*2}$, $\cdots$, $\mathbb{T}_{*v}$, which have the following properties:*

2.6.1 $\mathbb{T}_{i*} \cap \mathbb{T}_{j*} = \varnothing$ $(i \neq j)$; $\bigcup_{i=1}^{u} \mathbb{T}_{i*} = \mathbb{T}$; $\mathbb{T}_{*i} \cap \mathbb{T}_{*j} = \varnothing$ $(i \neq j)$; $\bigcup_{j=1}^{v} \mathbb{T}_{*j} = \mathbb{T}$.

2.6.2 $\mathbb{T}_{ij} = \mathbb{T}_{i*} \cap \mathbb{T}_{*j}$ *is either empty or consists of one commodity;*

2.6.3 If $\mathbb{T}_{ij} = \varnothing$, *then* $\mathbb{T}_{i,j+1} = \varnothing$;

2.6.4 For each set $\mathbb{T}_{i*}$ there is a commodity $s_i \in S$ such that if a process has β units of commodity in $\mathbb{T}_{i1}$ among its outputs, then it has β units of commodity s_i among its inputs;

2.6.5 Let $\mathbb{T}_{ij} \neq \varnothing$ with $j \geq 2$. If a process has β units of commodity in $\mathbb{T}_{ij}$ among its outputs, then it has β units of commodity in $\mathbb{T}_{i,j-1}$ among its inputs.

In the above assumption, u is the number of types of machines, and v is the number of ages of machines. Assumption 2.6 implies that all old machines can be divided into u groups according to their types, or into v groups according to their ages. In Assumption 2.6.2, commodity in $\mathbb{T}_{ij} \neq \varnothing$ is a j-year-old machine of type i. Assumption 2.6.3 states that if a j-year-old machine of type i does not exist, then no machine of type i can be $j+1$-year-old. Commodity s_i in Assumption 2.6.4 is a new machine. Assumption 2.6.5 implies that if there is β units j $(j \geq 2)$-year-old machines of type i produced by a process, then this process must use β units $j-1$-year-old machines of type i as an input.

In Salvadori (1999), one of the core assumptions is that the efficiency path of a transferable machine is independent of the sector in which the machine is used (called the Uniform Efficiency Path Axiom),[1] or formally:

Uniform Efficiency Path Axiom (Salvadori 1999). *If a type of machine is used in the production of finished goods i and j $(i \neq j)$, then there is a vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ such that for each process (a_s^T, b_s^T, l_s) producing finished good*

[1] For the definition of efficiency, see Kurz and Salvadori (1995, p. 203).

i using a machine of that type (old or new) there is a process (a_t^T, b_t^T, l_t) producing finished good j such that the vector (a_t^T, b_t^T, l_t) is a linear combination of vectors (a_s^T, b_s^T, l_s) and $(a_{ij}^T, b_{ij}^T, l_{ij})$.

This assumption guarantees that if old machines are not jointly utilised, and if we consider one transferable machine of type α involved in the production of finished good i and j $(i \neq j)$ as two different types of machines, then these two types of machines will have the same efficiency path (Salvadori, 1999). For instance, the efficiency paths of machines used to produce clothes of different colours (similar but different finished goods) will not be influenced by the production in which the machines are engaged.[2]

However, when old machines are jointly utilised, the Uniform Efficiency Path Axiom may not hold because of the following reasons. First, the vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ in the above assumption may not exist. This is because, for the non-jointly utilised case, the only differences between (a_s^T, b_s^T, l_s) and (a_t^T, b_t^T, l_t), which are normalised by the old machine of type α, are finished goods. This property may not hold when jointly utilised machines are allowed, since other types of machines of different ages may appear in (a_s^T, b_s^T, l_s) and (a_t^T, b_t^T, l_t). Thus the differences between (a_s^T, b_s^T, l_s) and (a_t^T, b_t^T, l_t) will include old machines of different ages and types, and it is impossible to find a vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ fulfilling the axiom. Secondly, when joint utilisation is allowed, the efficiency of one particular machine is not only determined by the circulating capital goods, labour input and output, but is also influenced by the other machines jointly utilised with it. In other words, the efficiencies of jointly utilised machines are interdependent. Since one transferable machine may be jointly utilised with different types of non-transferable machines in different sectors, the efficiency paths of this partic-

[2] The Uniform Efficiency Path Axiom may be misunderstood as the following meaning: the "efficiency" of a machine is independent of the finished goods it has produced because its efficiency is dependent on how many operations the machine has been used for. For instance, the efficiency of a saw is independent of the finished goods (chairs or tables) it has produced because its efficiency is dependent on how many timbers it has cut (the saw loses its sharpness due to repeated use). However, the efficiency of a machine, by definition (Kurz and Salvadori 1995, p. 203), is determined by the production and technique, which may be irrelevant to the operations. Therefore, the above idea may not be correct in general.

ular transferable machine may not be uniform, due to the influences of other non-transferable machines jointly utilised with it in different sectors. In sum, the axiom needs to be modified in order to deal with joint utilisation of old machines. The vector $(a_{ij}^T, b_{ij}^T, l_{ij})$ should correspond to the transferable machine of type α, and the Uniform Efficiency Path Axiom is maintained in the following way: the efficiency paths of the transferable machine of type α are uniform *if* other machines jointly utilised with it are neglected. In other words, the Uniform Efficiency Path Axiom should be modified as follows:

Assumption 2.7. *If a machine of type α is used in the production of finished goods i and j $(i \neq j)$, then there exists a vector $(a_{ij}^{\alpha T}, b_{ij}^{\alpha T}, l_{ij}^{\alpha})$ such that for each process (a_s^T, b_s^T, l_s) producing finished good i using a machine of type α (old or new), there is a process (a_t^T, b_t^T, l_t) producing finished good j such that $(a_t^T \mathbb{I}^\alpha, b_t^T \mathbb{I}^\alpha, l_t)$ is a linear combination of $(a_s^T \mathbb{I}^\alpha, b_s^T \mathbb{I}^\alpha, l_s)$ and $(a_{ij}^{\alpha T}, b_{ij}^{\alpha T}, l_{ij}^{\alpha})$, where $\mathbb{I}^\alpha$ is a diagonal matrix whose diagonal elements are either 1 or 0:*

$$\mathbb{I}_m^\alpha = \begin{cases} 1 & \textit{if either } m \in S \textit{ or } m \in H_i^\alpha \cap H_j^\alpha \\ 0 & \textit{otherwise} \end{cases}$$

$\mathbb{I}_m^\alpha$ is the diagonal element of $\mathbb{I}^\alpha$, and H_i^α means all machines of type α appearing in sector i either as an input or as an output.

Assumption 2.7 generalises the Uniform Efficiency Path Axiom because it allows jointly utilised machines (it is exactly the Uniform Efficiency Path Axiom if joint utilisation of old machines are excluded). On the other hand, it limits the explanation that the efficiency paths of transferable machines are uniform. This is due to the fact that the efficiencies of jointly utilised machines are interdependent. Hence, only in one of the following situations can Assumption 2.7 be explained as the fact that the transferable machines have uniform efficiency paths: (i) each transferable machine and other machines jointly utilised with it form a plant, i.e., they always appear in the processes together and with same age.[3]; (ii) transferable machines of different ages are jointly utilised with the same ages and same types of other machines;

[3]For the definition of plant, see Kurz and Salvadori (1995, p. 207, p. 266) In this case, the efficiency paths of the whole plant are uniform.

(iii) all other machines jointly utilised with transferable machines have constant efficiency; (iv) the transferable machine itself has constant efficiency. Assumption 2.7 will be called the modified Uniform Efficiency Path Axiom.

Now an example will be given to make the above assumptions clearer. Assume there are 7 commodities. The first 4 commodities belong to S; the others belong to $\mathbb{T}$. Consumption vector c has the following sign: $c = (+, +, 0, \cdots, 0)^T$. $u = 2$, $v = 2$, which means that there are 2 types of old machines, and the old machines are 2 years old. Label the types of machines as Π and Σ. $\mathbb{T}_{\Sigma 2} = \varnothing$, which means that there are no 2-year-old machines of type Σ or machine Σ lasts for 2 years. Commodity 3 is a new machine of type Π; commodity 4 is a new machine of type Σ. Some (not all) of the processes producing commodities 1 and 2 are listed in table 2.1.

All processes are normalised by the finished goods, i.e., $b_{11} = 1 = b_{72}$. From table 2.1 we can see that $\Pi \in K$ and $\Sigma \in H$. Based on the given processes in table 2.1, we know that there exist another four processes in sector 2 using the old machine Σ, if Assumption 2.7 holds. These processes are listed in table 2.2.

Since the non-transferable old machines jointly utilised with Σ are Π_1 and Π_2, Matrix $\mathbb{I}$ in Assumption 2.7 is:

$$\mathbb{I} = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 1 \end{bmatrix}$$

If Assumption 2.7 holds, there exists a vector $(a_{ij}^{\alpha T}, b_{ij}^{\alpha T}, l_{ij}^{\alpha})$ such that the following equations hold:

$$\text{Process}(7) \times \mathbb{I} = \text{Process}(1) \times \mathbb{I} + (a_{ij}^{\alpha T}, l_{ij}^{\alpha}, b_{ij}^{\alpha T})$$

$$\text{Process}(8) \times \mathbb{I} = \text{Process}(2) \times \mathbb{I} + (a_{ij}^{\alpha T}, l_{ij}^{\alpha}, b_{ij}^{\alpha T})$$

$$\cdots \quad \cdots$$

Or:

$$\begin{aligned} &(a_{11}, a_{12}, \pi_0, \sigma_0, 0, 0, 0, l_1, b_{11}, 0, 0, 0, 0, 0, \sigma_1) \\ -&(a_{71}, a_{72}, 0, \sigma_0, 0, 0, 0, l_7, 0, b_{72}, 0, 0, 0, 0, \sigma_1) \\ =\ &(a_{21}, a_{22}, \pi_0, 0, 0, 0, \sigma_1, l_2, b_{11}, 0, 0, 0, 0, 0, 0) \\ -&(a_{81}, a_{82}, 0, 0, 0, 0, \sigma_1, l_8, 0, b_{72}, 0, 0, 0, 0, 0) \end{aligned}$$

$$\cdots \quad \cdots$$

π_i and σ_i are the coefficients of machines (new and old) in the corresponding processes.[4] It should be emphasized that the processes listed in tables 2.1 and 2.2 are not *all* processes which are able to produce commodities 1 and 2. There may exist one process $8'$ which produces commodity 2 and uses old machines of the same types and the same ages as process 8, and

$$\text{Process}(8') \times \mathbb{I} \neq \text{Process}(2) \times \mathbb{I} + (a_{ij}^{\alpha T}, l_{ij}^{\alpha}, b_{ij}^{\alpha T})$$

but if Assumption 2.7 holds, we can find another process $2'$ such that:

$$\text{Process}(8') \times \mathbb{I} = \text{Process}(2') \times \mathbb{I} + (a_{ij}^{\alpha T}, l_{ij}^{\alpha}, b_{ij}^{\alpha T})$$

An intuitive explanation for the above example goes like this: there are two processes (1 and 2) using a pan (machine Π) and an oven (machine Σ) to produce biscuits (commodity 1). Another process (7) uses an oven (machine Σ) to produce cakes (commodity 2). When the oven is transferred from biscuit making to cake making (from process 1 to process 7), the change in output (b_{ij}^T) requires a change in the ingredients (a_{ij}^T) and labour (l_{ij}), while the pan (machine Π) is neglected. Therefore, it is reasonable to assume that, if we change the same ingredients (a_{ij}^T) and labour (l_{ij}) to process (2), and without using the pan (machine Π), we can get another process (8) which is

[4] In the same process, for instance, in process 1, $\pi_0 = \pi_1$, $\sigma_0 = \sigma_2$ because of the Assumption 2.6. Here different subscripts are used in order to make it clear that they are machines of different ages.

able to produce cakes.

In order to cover all the cases of fixed capital models and clarify the differences, additional assumptions will be given next.

Assumption 2.8. *There is only one type of machine $t \in \mathbb{T}$ that can appear in each process.*

Assumption 2.9. *Set $H = \varnothing$.*

In the non-transferable single machine fixed capital model, Assumptions 2.1 to 2.9 hold. In the non-transferable, jointly utilised machines fixed capital model, Assumptions 2.1 to 2.6, and 2.9 hold. In the transferable single machine fixed capital model, Assumptions 2.1 to 2.8 hold. In the transferable jointly utilised machines fixed capital model, which will be dealt in this chapter, Assumptions 2.1 to 2.7 hold.

2.3 Choice of Technique

This section will investigate the properties of the cost-minimising technique. One of the results of the jointly utilised machines model is that economic growth may affect choice of technique, but consumption does not (Salvadori 1988a, 1988b).[5] This chapter will show the same result when the modified Uniform Efficiency Path Axiom holds. Before looking at the main theorems, a few lemmas will be introduced.

There is said to exist a cost-minimising technique if there exists a solution (p^*, x^*, w^*) to the following system (Kurz and Salvadori (1995, chap. 8)):

$$[B - (1+r)A]p \leqq wl \tag{2.1a}$$
$$x^T[B - (1+r)A]p = wx^Tl \tag{2.1b}$$
$$x^T[B - A] \geqq d^T \tag{2.1c}$$
$$x^T[B - A]p = d^Tp \tag{2.1d}$$

[5] A similar result was obtained by Stigliz (1970) in a different approach, whose main purpose was to generalise the non-substitution theorem to the case with durable capital goods.

$$p \geqq 0, x \geqq 0, w \geqq 0, f^T p = 1 \tag{2.1e}$$

In the above equations, r is a given rate of profit, w is the rate of wage, x is the intensity vector, d is the requirement for use, p is the non-negative price vector, f is the semipositive numéraire whose elements are positive corresponding to the commodities which are certainly produced. Requirement for use d does not need to be constant: it can be a function of r, w, p, A, and B. In this chapter, d is assumed to have the following form:

$$d = gA^T x + c$$

where g is the uniform rate of growth which satisfies $g \leq r$, and c is the consumption vector, whose first s elements are non-negative and the last $n - s$ elements are zero.

System (2.1) is equivalent to the following system:

$$[B - (1 + r)A]y \leqq l \tag{2.2a}$$
$$q^T [B - (1 + r)A]y = q^T l \tag{2.2b}$$
$$q^T [B - (1 + g)A] \geqq c^T \tag{2.2c}$$
$$q^T [B - (1 + g)A]y = c^T y \tag{2.2d}$$
$$q \geqq 0, \quad y \geqq 0 \tag{2.2e}$$

If the following assumption holds, there exists a cost-minimising technique (Kurz and Salvadori 1995, chap. 8):

Assumption 2.10. *There exists a non-negative vector z such that the following inequality holds:*

$$z[B - (1 + r)A] \geqq c \tag{2.3}$$

Let Assumption 2.10 hold such that the cost-minimising technique exists. Let (q^*, y^*) be the solution to system (2.2). For (q^*, y^*), let the processes (A^*, B^*, l^*) be those such that inequality (2.2a) holds as an equation corresponding to y^*, i.e., the corresponding q_i^* to each process in (A^*, B^*, l^*) is positive. Order the processes such that the first m_1 processes produce finished good 1, and the following m_2 processes produce finished goods 2, and so on. The last m_s processes produce finished goods s. Without loss of generality, assume that A^* and B^* have the following form:

$$A^* = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} A_{11} & A_{1t_1} & A_{1t_2} & \cdots & A_{1t_u} \\ \hline A_{21} & A_{2t_1} & A_{2t_2} & \cdots & A_{2t_u} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline A_{s1} & A_{st_1} & A_{st_2} & \cdots & A_{st_u} \end{array}\right) \end{array} \tag{2.4}$$

$$B^* = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} B_{11} & B_{1t_1} & B_{1t_2} & \cdots & B_{1t_u} \\ \hline B_{21} & B_{2t_1} & B_{2t_2} & \cdots & B_{2t_u} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline B_{s1} & B_{st_1} & B_{st_2} & \cdots & B_{st_u} \end{array}\right) \end{array} \tag{2.5}$$

Denote t_j as the group of j type old machines, $j = 1, 2, \cdots u$. If $t_j \in K$, then only one sub-matrix A_{it_j} in A^* and only one sub-matrix B_{it_j} in B^* $(i = 1, 2, \cdots, s)$, are semipositive; all the others are zero. If $t_j \in H$, then at least two submatrices $A_{it_j}, A_{ht_j}(B_{it_j}, B_{ht_j})$ in $A^*(B^*)$ are semipositive.

We have:

$$[B^* - (1+r)A^*]y^* = l^* \tag{2.6}$$

$$\bar{q}^{*T}[B^* - (1+g)A^*] \geqq c^T \tag{2.7}$$

where $\bar{q}^*$ is obtained from q^* by deleting zero elements.

Lemma 2.1. *If there is a process producing finished good i and using an old machine of type α in (A^*, B^*, l^*), then there exists a process in (A^*, B^*, l^*) which produces finished good i using old machine of type α as an input but without producing old machine of type α. In addition, if the age of the α type machine in the latter process is ρ, then (A^*, B^*, l^*) must include processes producing finished good i using the α type machine of age $1, 2, \cdots, \rho$.*

Proof.

Suppose that the α type machine of age τ ($\tau = 1, 2, \cdots, \rho$) is produced by the process producing finished good i in (A^*, B^*, l^*). If there is no process using the α type machine of age τ as an input, then the α type machine of age τ is overproduced, which means it has zero price. Hence, there exists another process with the same inputs and the same outputs, except the α type machine of age τ is not produced (Assumption 2.4.4) in (A^*, B^*, l^*), or there exists a process using the old machine of type α as an input but without producing the old machine of type α.

- If the process using the α type old machine of age τ as an input is in sector i, then the lemma is proved.

- If the process mentioned is in sector j ($j \neq i$), then there exists a process using the same age of old machine of type α as an input in sector i. Suppose the opposite, that is there is no such process in sector i.

  Denote the process producing finished good i and producing the α type old machine of age τ as $e_i^T(A^*, B^*, l^*)$, and denote the process producing finished good j that uses the α type old machine of age τ as input as $e_j^T(A^*, B^*, l^*)$. Then there will be another two processes $e_u^T(A, B, l)$ and $e_v^T(A, B, l)$, which produce finished good i and j respectively, and which have the same inputs and outputs (including other old machines) as $e_i^T(A^*, B^*, l^*)$ and $e_j^T(A^*, B^*, l^*)$ except old machine of type α. $e_u^T(A, B, l)$ uses the same age of α type old machine as $e_j^T(A^*, B^*, l^*)$, and $e_v^T(A, B, l)$ produces the same age of α type old machine as $e_i^T(A^*, B^*, l^*)$.

  Since $e_i^T(A^*, B^*, l^*)$ and $e_j^T(A^*, B^*, l^*)$ are in (A^*, B^*, l^*) while $e_u^T(A, B, l)$ is not, we have:

$$e_i^T[B^* - A^*(1+r)]y^* = e_i^T l^* \tag{2.8}$$

$$e_j^T[B^* - A^*(1+r)]y^* = e_j^T l^* \tag{2.9}$$

$$e_u^T[B - A(1+r)]y^* < e_u^T l \tag{2.10}$$

According to Assumption 2.7, we have:

$$\begin{aligned} &(e_v^T A\mathbb{I}^\alpha, e_v^T B\mathbb{I}^\alpha, e_v^T l) - (e_i^T A^*\mathbb{I}^\alpha, e_i^T B^*\mathbb{I}^\alpha, e_i^T l^*) \\ =\ &(e_j^T A^*\mathbb{I}^\alpha, e_j^T B^*\mathbb{I}^\alpha, e_j^T l^*) - (e_u^T A\mathbb{I}^\alpha, e_u^T B\mathbb{I}^\alpha, e_u^T l) \end{aligned} \tag{2.11}$$

This is because the processes are already normalised by the finished goods that they produce. Since $e_u^T(A, B, l)$ and $e_i^T(A^*, B^*, l^*)$ ($e_v^T(A, B, l)$ and $e_j^T(A^*, B^*, l^*)$) use the same ages and same types of other old machines, we have:

$$e_v^T(A, B, l) = e_i^T(A^*, B^*, l^*) + [e_j^T(A^*, B^*, l^*) - e_u^T(A, B, l)] \tag{2.12}$$

which means:

$$e_v^T[B - A(1+r)]y^* > e_v^T l \tag{2.13}$$

This is a contradiction, since y^* is a solution to system (2.2). Therefore, $e_u^T(A, B, l)$ must be in (A^*, B^*, l^*).

Since $\tau = 1, 2, \cdots, \rho$, (A^*, B^*, l^*) includes processes producing finished good i and using old machines of type α with ages of $1, 2, \cdots, \rho$.

Q.E.D.

Now an example will be given to make the lemma clearer. We still use the example in table 2.1 and table 2.2. Assume that processes 1 and 8 are in (A^*, B^*, l^*) while processes 2 and 7 are not. We have:

$$(a_{11}y_1^* + a_{12}y_2^* + \pi_0 y_{\pi_0}^* + \sigma_0 y_{\sigma_0}^*)(1+r) + l_1 = b_{11}y_1^* + \pi_1 y_{\pi_1}^* + \sigma_1 y_{\sigma_1}^* \tag{2.14a}$$

$$(a_{81}y_1^* + a_{82}y_2^* + \sigma_1 y_{\sigma_1}^*)(1+r) + l_8 = b_{72}y_2^* \tag{2.14b}$$

Table 2.1: Some Processes Producing Commodities 1 and 2 of the Example

	Inputs									Outputs						
	1	2	Π_0	Σ_0	Π_1	Π_2	Σ_1	L		1	2	Π_0	Σ_0	Π_1	Π_2	Σ_1
Process (1)	a_{11}	a_{12}	+	+	0	0	0	l_1	$\rightarrow$	b_{11}	0	0	0	+	0	+
Process (2)	a_{21}	a_{22}	+	0	0	0	+	l_2	$\rightarrow$	b_{11}	0	0	0	+	0	0
Process (3)	a_{31}	a_{32}	0	+	+	0	0	l_3	$\rightarrow$	b_{11}	0	0	0	0	+	+
Process (4)	a_{41}	a_{42}	0	0	+	0	+	l_4	$\rightarrow$	b_{11}	0	0	0	0	+	0
Process (5)	a_{51}	a_{52}	0	+	0	+	0	l_5	$\rightarrow$	b_{11}	0	0	0	0	0	+
Process (6)	a_{61}	a_{62}	0	0	0	+	+	l_6	$\rightarrow$	b_{11}	0	0	0	0	0	0
Process (7)	a_{71}	a_{72}	0	+	0	0	0	l_7	$\rightarrow$	0	b_{72}	0	0	0	0	+
Process (8)	a_{81}	a_{82}	0	0	0	0	+	l_8	$\rightarrow$	0	b_{72}	0	0	0	0	0

$$(a_{21}y_1^* + a_{22}y_2^* + \pi_0 y_{\pi_0}^* + \sigma_1 y_{\sigma_1}^*)(1+r) + l_2 > b_{11}y_1^* + \pi_1 y_{\pi_1}^* \quad (2.14c)$$

If Assumption 2.7 holds, add equation (2.14a) to equation (2.14b), and subtract inequality (2.14c), we have:

$$(a_{71}y_1^* + a_{72}y_2^* + \sigma_0 y_{\sigma_0}^*)(1+r) + l_7 < b_{72}y_2^* + \sigma_1 y_{\sigma_1}^* \quad (2.15)$$

which is a contradiction since y^* is a solution to system (2.2). Therefore processes 2 and 7 must be in (A^*, B^*, l^*).

Lemma 2.1 implies that if there exists a machine of type α which lasts for $\rho+1$ years in the cost-minimising technique, and is used in the processes producing finished goods i and j ($i \neq j$), then there are $\rho+1$ processes producing finished good i using the machine of type α from age 0 to ρ, and $\rho+1$ processes producing finished good j using the machine of type α from age 0 to ρ. In this situation, ρ processes will be a linear combination of others in order to eliminate overdetermination of prices, which further implies that the $\rho+1$ processes producing finished good i and using the machine of type α use the same age and same type of other old machines (if there are any jointly utilised with the α type machine) except the machine of type α. The same condition applies to the processes producing finished good j using the machine of type α.

Table 2.2: Processes Fulfilling Assumption 2.7 in Sector 2 of the Example

	Inputs									Outputs						
	1	2	Π_0	Σ_0	Π_1	Π_2	Σ_1	L		1	2	Π_0	Σ_0	Π_1	Π_2	Σ_1
Process (9)	a_{91}	a_{92}	0	+	0	0	0	l_9	$\rightarrow$	0	b_{72}	0	0	0	0	+
Process (10)	$a_{10,1}$	$a_{10,2}$	0	0	0	0	+	l_{10}	$\rightarrow$	0	b_{72}	0	0	0	0	0
Process (11)	$a_{11,1}$	$a_{11,2}$	0	+	0	0	0	l_{11}	$\rightarrow$	0	b_{72}	0	0	0	0	+
Process (12)	$a_{12,1}$	$a_{12,2}$	0	0	0	0	+	l_{12}	$\rightarrow$	0	b_{72}	0	0	0	0	0

Lemma 2.2. *For (A^*, B^*, l^*), there exists semipositive $\hat{q}^* = [\hat{q}_1^{*T}, \hat{q}_2^{*T}, \cdots, \hat{q}_s^{*T}]^T$, which satisfies $\hat{q}^* = \lambda\bar{q}^*$, such that:*

$$\hat{q}_i^{*T}[B_{it_j} - (1+g)A_{it_j}] \geqq 0 \tag{2.16}$$

$$\hat{q}_i^{*T} B_{i1} = e_i^T \tag{2.17}$$

$$i = 1, 2, \cdots, s; j = 1, 2, \cdots, u.$$

Proof.

Without loss of generality, let us prove that there exists $\hat{q}_1^*$ for sector 1 which satisfies the lemma.

Assume that the old machine t_i lasts for τ_i years, and let t_{ij} denote the j-year-old machine of type i used in sector 1, $j = 1, \cdots, \tau_i - 1$, $i = 1, 2, \cdots, u$.

CASE 1. If $t_{ij} \in K$ for all i, then $\hat{q}_1^* = \lambda\bar{q}_1^*$ such that $\hat{q}_1^{*T} B_{11} = e_1^T$ satisfies the lemma, where $\bar{q}_1^*$ is the first m_1 elements of $\bar{q}^*$.

CASE 2. If for some i, $t_{ij} \in H$, since Lemma 2.1 holds, there exist processes using transferable machine t_i from age 0 to $\tau_i - 1$, and these processes use the same types and same ages of other machines (if they are jointly utilised with t_i). Hence, we can reorder the processes which use transferable machines in the following way:

The first process uses finished goods (including all transferable new machines used in sector 1) and non-transferable machines jointly utilised with first transferable machine t_1 to produce finished good 1, all one-year-old transferable machines utilised in sector 1 and other non-transferable machines. The second process uses finished goods, one-year-old transferable old machine t_1 and other old machines of the same type and age as the first process to produce finished good 1, two-year-old machine t_1 and all the other old machines, and so on until the (τ_1)th process which uses finished goods and $\tau_1 - 1$-year-old transferable machine t_1 and other old machines of the same type and same age as the first process, to produce finished good 1 and all other machines but without producing old machine t_1. Then the $(\tau_1 + 1)$th process uses finished goods and one-year-old transferable machine t_2 and machines of the same types and ages as process 1 to produce finished good 1, one-year-old machine t_1, two-year-old transferable machine t_2, and all other machines, and so on. We list all these processes in table 2.3 (a_{is} and e_{1s} are used to denote the subvectors of inputs and outputs of finished goods in process i, respectively).

Denote the corresponding intensities of these processes as q'_1 (obtained from $\bar{q}^*$). We can find another intensity vector q''_1 such that at this intensity, these processes are "equivalent" to an integrated process which only uses non-transferable machines. Normalise q''_1 such that $e^T q'_1 = e^T q''_1$, which means the intensities will not change the types, ages and total quantities of non-transferable machines. One way to prove the existence of q''_1 is to use a similar procedure to Sraffa's (1960): consider the processes from 1 to τ_1 and denote them as $(A_{(t_1,t_{-10})}, B_{(t_1,t_{-10})}, l_{(t_1,t_{-10})})$, and normalise transferable machine t_1 as 1. We can find a vector $q'_{11} = [(1+g)^{\tau_1 - 1}, (1+g)^{\tau_1 - 2}, \cdots, (1+g), 1]^T$ such that $q'^T_{11}[B_{(t_1,t_{-10})} - (1+g)A_{(t_1,t_{-10})}]$ is "equivalent" to a process without using t_1. Continue the procedure for the next τ_1 processes; we will get another "equivalent" process without using old machines of type 1. The same procedure can be applied to these "equivalent" processes. Hence, in the end we will find the q''_1 as we want. Or more formally, we can use the Alternative Theorem (Gale, 1960, p. 48) to prove the existence q''_1, which is presented in the Appendix.

The existence of q''_1 implies that there exists an intensity vector such

Table 2.3: All Processes Using the Transferable Machines in Sector 1 of Lemma 2.2

	Inputs							Outputs				
	S	T_1	T_2	$\cdots$	T_u	L		S	T_1	T_2	$\cdots$	T_u
(1)	a^T_{1s}	0	0	$\cdots$	t_{uj}	l_1	$\rightarrow$	e^T_{1s}	t_{11}	t_{21}	$\cdots$	$t_{u,j+1}$
(2)	a^T_{2s}	t_{11}	0	$\cdots$	t_{uj}	l_2	$\rightarrow$	e^T_{1s}	t_{12}	t_{21}	$\cdots$	$t_{u,j+1}$
						$\vdots$	$\vdots$					
(τ_1)	$a^T_{\tau_1,s}$	$t_{1(\tau_1-1)}$	0	$\cdots$	t_{uj}	l_{τ_1}	$\rightarrow$	e^T_{1s}	0	t_{21}	$\cdots$	$t_{u,j+1}$
(τ_1+1)	$a^T_{\tau 1+1,s}$	0	t_{21}	$\cdots$	t_{uj}	l_{τ_1+1}	$\rightarrow$	e^T_{1s}	t_{11}	t_{22}	$\cdots$	$t_{u,j+1}$
(τ_1+2)	$a^T_{\tau 1+2,s}$	t_{11}	t_{21}	$\cdots$	t_{uj}	l_{τ_1+2}	$\rightarrow$	e^T_{1s}	t_{12}	t_{22}	$\cdots$	$t_{u,j+1}$
						$\vdots$	$\vdots$					
$(\tau_1+\tau_1)$	$a^T_{\tau_1+\tau_1,s}$	$t_{1(\tau_1-1)}$	t_{21}	$\cdots$	t_{uj}	$l_{\tau_1+\tau_1}$	$\rightarrow$	e^T_{1s}	0	t_{22}	$\cdots$	$t_{u,j+1}$
						$\vdots$	$\vdots$					

that we get an equivalent "integrated" process without using transferable machines in sector 1. Then we arrive at the same situation as CASE 1. Hence $\hat{q}^*_1$ exists.

Q.E.D.

We still use the example in table 2.1 and table 2.2 to show the existence of $\hat{q}^*_1$ in Lemma 2.2. In the above example, Π is a non-transferable machine, and Σ is a transferable machine. Assume that the processes 1, 3, 5, 7 and 8 are in (A^*, B^*, l^*). According to Lemma 2.1, there exist other processes producing commodity 1 and using the same ages of type Π as processes 1, 3, 5 in (A^*, B^*, l^*). Assume that these processes are 2, 4 and 6. Without loss of generality, assume all these processes use one unit machine Σ. Let us prove the existence of $\hat{q}^*_1$ by using Sraffa's approach. It can be checked that if the processes 1 to 6 are operated at the intensity $q''_1 = \lambda(1+g, 1, 1+g, 1, 1+g, 1)^T$, where λ is an appropriate scalar, we can get "integrated" processes which only use the non-transferable machine Π. Then we arrive at the same situation as CASE 1 in Lemma 2.2, and the existence of $\bar{q}^*$ (solution to system (2.2)) thus guarantees that $\hat{q}^*_1$ exists.

Now we can prove that the consumption does not influence choices of technique.

Theorem 2.1. *If Assumptions 2.1 to 2.7, and Assumption 2.10 hold, there exists a solution* (q^*, y^*) *to system* (2.2) *for* $c = c_1$. *Then there exists a solution* (q^{**}, y^*) *to system* (2.2) *for* $c = c_2$. *In addition, the first* s *elements of* y^*, *denoted by* y_s^*, *are positive.*

Proof.

Let $B^* = (B_s^*, B_t^*)$, $A^* = (A_s^*, A_t^*)$, $y^* = (y_s^{*T}, y_t^{*T})^T$, where B_s^* and A_s^* are formed by the first s columns of B^* and A^*, respectively.

Since Lemma 2.2 holds, there exists a semipositive matrix $Q(g)$:

$$Q(g) = \begin{bmatrix} q_1^{*T} & & & \\ & q_2^{*T} & & \\ & & \ddots & \\ & & & q_s^{*T} \end{bmatrix}$$

such that $QB^*[I, 0]^T = QB_s^* = I$, where I is an $s \times s$ identical matrix, and the following relations hold:

$$e^T Q[B_s^* - (1+g)A_s^*] \geqq c_{1s} \tag{2.18}$$

$$Q[B_t^* - (1+g)A_t^*] \geqq 0 \tag{2.19}$$

c_{1s} is the first s elements of c_1. Equation (2.18) means that for the square matrix $Q[B_s^* - (1+g)A_s^*]$, there exists a positive vector e such that $e^T Q[B_s^* - (1+g)A_s^*]$ is semipositive. Therefore $Q[B_s^* - (1+g)A_s^*]$ is invertible and the inverse is semipositive (Kurz and Salvadori 1995, p. 510-11, Theorem A.3.1). Hence, there exists a non-negative vector v such that:

$$v^T Q[B_s^* - (1+g)A_s^*] = c_{2s} \tag{2.20}$$

$$v^T Q[B_t^* - (1+g)A_t^*] \geqq 0 \tag{2.21}$$

where c_{2s} is the first s elements of c_2. Let $\bar{q}^{**} = (v_1 q_1^{*T}, \cdots, v_i q_i^{*T} \cdots, v_s q_s^{*T})^T$, v_i is the ith element of v. From equation (2.20) and inequality (2.21) we know that:

$$\bar{q}^{**T}[B^* - (1+g)A^*] \geqq c_2 \tag{2.22}$$

i.e., there exists a solution (q^{**}, y^*) to system (2.2) for $c = c_2$, q^{**} is $\bar{q}^{**}$ augmented with zeros.

Since:

$$\bar{q}^{*T}[B^* - (1+r)A^*]y^* = \bar{q}^{*T}l^* \tag{2.23}$$

$$\bar{q}^{*T}[B^* - (1+g)A^*]y^* = c^T y^* \tag{2.24}$$

Therefore:

$$Q[B^*-(1+g)A^*]y^* = Q[B^*-(1+r)]y^*+(r-g)QA^*y^* = Ql^*+(r-g)QA^*y^* \geq 0 \tag{2.25}$$

$$Q[B_t^* - (1+g)A_t^*]y_t^* = 0 \tag{2.26}$$

Hence:

$$Q[B_s^*-(1+g)A_s^*]y_s^* = Q[B^*-(1+g)A^*]y^*-Q[B_t^*-(1+g)A_t^*]y_t^* \geqq Ql^* \tag{2.27}$$

Because $Q[B_s^* - (1+g)A_s^*]$ is invertible and the inverse is semipositive, and inequality (2.19) holds, we have:

$$\Delta \equiv \{Q[B_s^* - (1+g)A_s^*]\}^{-1}Q[B^* - (1+g)A^*] \geqq 0^T \tag{2.28}$$

Note that the sub-matrix formed by the first s columns of Δ is the identity matrix. Hence whatever is i ($i = 1, 2, \cdots, s$):

$$e_i^T\{Q[B_s^* - (1+g)A_s^*]\}^{-1}Q[B^* - (1+g)A^*] \geq 0^T \tag{2.29}$$

Since $e_i^T\{Q[B_s^* - (1+g)A_s^*]\}^{-1}Q \geqq 0$, and according to Assumption 2.3 and inequality (2.29), we have:

$$e_i^T\{Q[B_s^* - (1+g)A_s^*]\}^{-1}Ql^* > 0 \tag{2.30}$$

whatever is i, or:

$$\{Q[B_s^* - (1+g)A_s^*]\}^{-1} Ql^* > 0 \tag{2.31}$$

Therefore, y_s^* is positive because of inequalities (2.27) and (2.31).

Q.E.D.

It is also possible to prove that under the condition of $g = r$, if more than one cost-minimising technique exists, the prices of finished goods in terms of the wage rate are unique.

Theorem 2.2. *If $g = r$, and (q^*, y^*) and (q'^*, y'^*) are two solutions to system (2.2), then $y_s^* = y_s'^*$.*[6]

Proof.

Let (A^*, B^*, l^*) and (A'^*, B'^*, l'^*) be the processes corresponding to y^* and y'^* which satisfy equation (2.6), respectively.

Then we have:

$$[B^* - (1+r)A^*]y'^* \leqq l^* \tag{2.32}$$

$$[B'^* - (1+r)A'^*]y^* \leqq l'^* \tag{2.33}$$

There exist $Q^*(r)$ and $Q'^*(r)$ defined as Theorem 2.1 because of Lemma 2.2. From equations (2.32) and (2.33) we have:

$$Q^*(r)[B^* - (1+r)A^*]y'^* \leqq Q^*(r)l^* = Q^*(r)[B^* - (1+r)A^*]y^* \tag{2.34}$$

$$Q'^*(r)[B'^* - (1+r)A'^*]y^* \leqq Q'^*(r)l'^* = Q'^*(r)[B'^* - (1+r)A'^*]y'^* \tag{2.35}$$

Since inequality (2.19) and equation (2.26) hold, inequality (2.34) implies:

$$Q^*(r)[B_s^* - (1+r)A_s^*]y_s'^* \leqq Q^*(r)[B_s^* - (1+r)A_s^*]y_s^* \tag{2.36}$$

[6] It should be noted that this theorem may not hold if $g \neq r$, see Salvadori (1988a).

From Theorem 2.1 we know that $Q^*(r)[B_s^* - (1+r)A_s^*]$ is invertible and the inverse is semipositive, and multiply the inverse on both sides of inequality (2.36):

$$y_s'^* \leqq y_s^* \tag{2.37}$$

From equation (2.35) we will have $y_s^* \leqq y_s'^*$, and therefore $y_s^* = y_s'^*$.

Q.E.D.

2.4 Further Discussions on the Modified Uniform Efficiency Path Axiom

From the above section we can see that, when transferable machines are allowed, the modified Uniform Efficiency Path Axiom, or Assumption 2.7, is crucial in order for the cost-minimising technique to be independent of consumption and other theorems. If the modified Uniform Efficiency Path Axiom does not hold, the system may be "interlocked" (Schefold 1989, p. 160). In such a situation, different sectors will be linked together through the transferable machines used in these sectors, and pure joint production appears. For instance, suppose the modified Uniform Efficiency Path Axiom does not hold, and one transferable machine of type α appears in the processes producing finished goods i and j $(i \neq j)$. Further, suppose in the cost-minimising technique, a one-year-old machine of type α is produced by one process producing finished good i, and the process using the one-year-old machine of type α as an input produces finished good j, but not i (i.e., lemma 2.1 fails). Now suppose the consumption changes but the cost-minimising technique does not. The intensities of activated processes of sector i and j will then be adjusted in order to fulfil the new consumption. However, during this adjustment, it is likely that the input of the one-year-old machine of type α required by sector j will be more than the one-year-old machine of type α produced by sector i, i.e., a negative quantity of one-year-old machine of type α will occur. Hence, a change in technique may be necessary when consumption changes.

In order to ensure that the cost-minimising technique is independent of consumption, what is needed is to break the "interlock". The Uniform Efficiency Path Axiom is an effective tool for this purpose. In this section, it will be shown that this assumption is not only sufficient, but also necessary.

Theorem 2.3. *If Assumptions 2.1 to 2.6, and Assumption 2.10 hold, and* (q^*, y^*) *is a solution to system* (2.2) *such that* y^* *is independent of* c*, then the modified Uniform Efficiency Path Axiom, i.e., Assumption 2.7 holds.*

Proof.

Without loss of generality, consider that one transferable machine of type α, which lasts $\tau_\alpha + 1$ years, appears in two different sectors i and j. Let matrices $(A^\alpha_{ij}, B^\alpha_{ij}, l^\alpha_{ij})$ be all the processes using the machine of type α in sectors i and j, and order the first m_i processes as those producing finished good i using the machine of type α from age 1 to age τ_α, and the following m_j processes producing finished good j are listed in the same way.

$$A^\alpha_{ij} = \begin{array}{c} \\ m_i \\ m_j \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} A^\alpha_{i1} & A^\alpha_{it_1} & A^\alpha_{it_2} & \cdots & A^\alpha_{it_u} \\ \hline A^\alpha_{j1} & A^\alpha_{jt_1} & A^\alpha_{jt_2} & \cdots & A^\alpha_{jt_u} \end{array}\right) \end{array} = \left(\begin{array}{c} A^\alpha_i \\ \hline A^\alpha_j \end{array}\right)$$

$$B^\alpha_{ij} = \begin{array}{c} \\ m_i \\ m_j \end{array} \begin{array}{c} \begin{array}{ccccc} s & t_1 & t_2 & \cdots & t_u \end{array} \\ \left(\begin{array}{c|c|c|c|c} B^\alpha_{i1} & B^\alpha_{it_1} & B^\alpha_{it_2} & \cdots & B^\alpha_{it_u} \\ \hline B^\alpha_{j1} & B^\alpha_{jt_1} & B^\alpha_{jt_2} & \cdots & B^\alpha_{jt_u} \end{array}\right) \end{array} = \left(\begin{array}{c} B^\alpha_i \\ \hline B^\alpha_j \end{array}\right)$$

$$l^\alpha_{ij} = \begin{pmatrix} l^\alpha_i \\ l^\alpha_j \end{pmatrix}$$

For given g and r $(g \leq r)$, suppose the machine of type α lasts for ρ $(\rho \leq \tau_\alpha)$ years in the cost-minimising technique. Therefore, $\rho + 1$ processes in $(A^\alpha_{ij}, B^\alpha_{ij}, l^\alpha_{ij})$ are enough to determine the prices of finished goods i, j and old machines of type α of different ages. However, since the cost-minimising technique is independent of consumption, i.e., a change in c will not influence y^* as a solution to system (2.2), "interlock" cannot exist. Therefore there must be ρ processes producing finished good i using the machine of type α from age 0 to age $\rho - 1$, and ρ processes producing finished good j using the machine of type α from age 0 to age $\rho - 1$, i.e., there must be 2ρ processes in the cost-minimising technique.

In order to avoid overdetermination of prices, $\rho - 1$ processes must be linear combinations of other $\rho+1$ processes. Let the$(\rho+t)$th process $(a^{jT}_{\rho+t}, b^{jT}_{\rho+t}, l^{j}_{\rho+t})$ producing finished good j be the linear combination of first $\rho + 1$ processes, $1 < t \leq \rho$. Since the $(\rho + 1)$th process $(a^{jT}_{\rho+1}, b^{jT}_{\rho+1}, l^{j}_{\rho+1})$ uses the α type machine of the same age as the first process $(a^{iT}_{1}, b^{iT}_{1}, l^{i}_{1})$, and the $(\rho + t)$th process $(a^{jT}_{\rho+t}, b^{jT}_{\rho+t}, l^{j}_{\rho+t})$ uses the α type machine of the same age as the tth process $(a^{iT}_{t}, b^{iT}_{t}, l^{i}_{t})$, the $(\rho+t)$th process $(a^{jT}_{\rho+t}, b^{jT}_{\rho+t}, l^{j}_{\rho+t})$ can only be the linear combination (with positive coefficients) of $(a^{iT}_{1}, b^{iT}_{1}, l^{i}_{1})$, $(a^{iT}_{t}, b^{iT}_{t}, l^{i}_{t})$ and $(a^{jT}_{\rho+1}, b^{jT}_{\rho+1}, l^{j}_{\rho+1})$, or:

$$(a^{jT}_{\rho+t}, b^{jT}_{\rho+t}, l^{j}_{\rho+t}) = (a^{iT}_{t}, b^{iT}_{t}, l^{i}_{t}) - (a^{iT}_{1}, b^{iT}_{1}, l^{i}_{1}) + (a^{jT}_{\rho+1}, b^{jT}_{\rho+1}, l^{j}_{\rho+1})$$

The above equation holds for all t. Hence it requires that: (i) other old machines jointly utilised with α type in the above processes in sector i must be the same type and the same age, or must have constant efficiencies, and the same requirements apply to the above processes in sector j. (ii) There must exist a vector $(a^{\alpha T}_{ij}, b^{\alpha T}_{ij}, l^{\alpha}_{ij})$ such that:

$$(a^{\alpha T}_{ij}, b^{\alpha T}_{ij}, l^{\alpha}_{ij}) = (a^{jT}_{\rho+t}\mathbb{I}^{\alpha}, b^{jT}_{\rho+t}\mathbb{I}^{\alpha}, l^{j}_{\rho+t}) - (a^{iT}_{t}\mathbb{I}^{\alpha}, b^{iT}_{t}\mathbb{I}^{\alpha}, l^{i}_{t}) \tag{2.38}$$

holds for all t, where $\mathbb{I}^{\alpha}$ is defined as Assumption 2.7. When g changes, the cost-minimising technique may change. Therefore equation (2.38) must hold for all processes in $(A^{\alpha}_{ij}, B^{\alpha}_{ij}, l^{\alpha}_{ij})$, which means that Assumption 2.7 is a necessary condition for the cost-minimising technique to be independent of consumption.

Q.E.D.

2.5 Conclusion

This chapter investigated the fixed capital model with both transferable and jointly utilised machines. Based on the assumptions in the chapter, it is shown that the choice of technique is independent of consumption, though it might be influenced by economic growth. Further, under the condition of $g = r$, the prices in terms of the wage rate of finished goods are unique even if there is more than one cost-minimising technique. It is also shown that if the cost-minimising technique is not affected by consumption, the system cannot be interlocked. The modified Uniform Efficiency Path Axiom is an effective tool to break the interlock (a sufficient condition), and is also a necessary condition for this "good" property.

Appendix: Proof of existence of q_1'' in Lemma 2.2

Suppose there are j-type transferable machines. Choose all the columns corresponding to transferable machines in the above processes and label the matrix as C. C has $\prod_{i=1}^{j} \tau_i$ rows and $\sum_{i=1}^{j}(\tau_i - 1)$ columns. We are looking for a q_1'' such that:

$$q_1''^T C = 0 \tag{2.39}$$

If we cannot find a solution to the following inequality, then the q_1'' exists:

$$Cz > 0 \tag{2.40}$$

where z is a $\sum_{i=1}^{j}(\tau_i - 1)$ vector.

Let us label the elements of z on the top of C (using $\kappa \equiv \sum_{i=1}^{j}(\tau_i - 1)$ for short), and call the first τ_1 row group 1 (denote by η_1), and so on. C has the following form:

$$C = \begin{array}{c|ccccc|cccc|c|cccc|}
 & z_1 & z_2 & \cdots & \cdots & z_{\tau_1-1} & z_{\tau 1} & \cdots & \cdots & \cdots & \cdots & z_{\kappa-\tau_u} & \cdots & z_{\kappa-1} & z_\kappa \\
 & + & & \cdots & & & + & & \cdots & & \cdots & + & & \cdots & \\
 & - & + & \cdots & & & + & & \cdots & & \cdots & + & & \cdots & \\
\eta_1 & & & \cdots & & & \vdots & & \cdots & & \vdots & \vdots & & \cdots & \\
 & & & \cdots & - & + & + & & \cdots & & \cdots & + & & \cdots & \\
 & & & \cdots & & - & + & & \cdots & & \cdots & + & & \cdots & \\
\hline
 & + & & \cdots & & & - & + & \cdots & & \cdots & + & & \cdots & \\
 & - & + & \cdots & & & - & + & \cdots & & \cdots & + & & \cdots & \\
\eta_2 & & & \cdots & & & \vdots & & \cdots & & \vdots & \vdots & & \cdots & \\
 & & & \cdots & - & + & - & + & \cdots & & \cdots & + & & \cdots & \\
 & & & \cdots & & - & - & + & \cdots & & \cdots & + & & \cdots & \\
\hline
 & & & \cdots & & & & & \cdots & & \cdots & & & \cdots & \\
\hline
 & + & & \cdots & & & & & \cdots & - & \cdots & & & \cdots & - \\
 & - & + & \cdots & & & & & \cdots & - & \cdots & & & \cdots & - \\
\eta_m & & & \cdots & & & & & \cdots & \vdots & \vdots & & & \cdots & \vdots \\
 & & & \cdots & - & + & & & \cdots & - & \cdots & & & \cdots & - \\
 & & & \cdots & & - & & & \cdots & - & \cdots & & & \cdots & -
\end{array}$$

Let $C = \Lambda\bar{C}$, where Λ is a semipositive diagonal matrix, such that for each column of $\bar{C}$, the absolute values of negative elements are larger than or equal to those of the positive elements (equality holds for $g = 0$).[7] $\bar{C}$ has the same structure as C and if we cannot find a $\bar{z}$ such that $\bar{C}\bar{z} > 0$, then we cannot find a z which satisfies inequality (2.40).

Now we will prove that there is no $\bar{z}$ (and z) satisfying inequality (2.40). Suppose the opposite and such a $\bar{z}$ exists.

First, for each column of $\bar{C}$, the absolute values of negative elements are larger than or equal to the positive elements. Hence we have $e^T\bar{C} \leqq 0$ (the equality holds when growth rate $g = 0$). Since $\bar{z}$ exists and $\bar{C}\bar{z} > 0$, then $e^T\bar{C}\bar{z} > 0$. If $g = 0$ we have $e^T\bar{C}\bar{z} = 0$, which is a contradiction. Therefore we just consider $g > 0$ and $e^T\bar{C} < 0$. $e^T\bar{C}\bar{z} > 0$ implies that some of $\bar{z}_i$, $i = 1, \cdots, \kappa$ must be negative.

Secondly, $\bar{z}$ cannot be negative. At least the first row of $\bar{C}$ requires that $\bar{z}_1, \bar{z}_{\tau_1}, \cdots, \bar{z}_{\kappa-\tau_u}$ cannot be negative simultaneously. Hence, without loss of generality, assume the last $\kappa - i$ elements of $\bar{z}$ are negative, $1 \leqq i < \kappa$, others are semipositive and at least one positive.

Thirdly, (i) if $1 < i \leqq \tau_1 - 1$, the submatrix formed by the first $i + 1$ rows of η_1, denoted by $\bar{C}_1$, has the property that the first i elements of $e^T\bar{C}_1$ are negative while the remaining elements are semipositive. Hence $e^T\bar{C}_1\bar{z} \leq 0$ because of the signs of elements of $\bar{z}$, which contradicts the fact that $\bar{C}_1\bar{z} > 0$ and $e^T\bar{C}_1\bar{z}$ is positive. (ii) If $\tau_1 - 1 < i < \kappa$, then the submatrix $\bar{C}_2$ formed by the first j rows of $\bar{C}$ such that $\bar{c}_{ji} < 0$ and $\bar{c}_{j+1,i} = 0$, has the property that the first i elements of $e^T\bar{C}_2$ are negative while the remaining are semipositive. Hence $e^T\bar{C}_2\bar{z} \leq 0$ and the same contradiction appears.

In sum, there is no $\bar{z}$ and z that can satisfy the inequality (2.40), and the Alternative Theorem guarantees that there exists a semipositive vector q_1'' satisfying equation (2.39).

[7] Matrix Λ exists because of the following reasons. In each group, for columns z_1 to z_{τ_1-1} the absolute values of negative elements are larger than or equal to the positive elements (since transferable machine t_1 is normalised.). If, for instance, the absolute values of negative elements are smaller than the positive elements in column z_{τ_1}, then a multiplication to all rows in group 2 by a large enough positive number λ_2 will get the result we want, and this will not change the properties of elements for columns z_1 to z_{τ_1-1} in group 2.

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Chapter 3

A Simple Generalisation of the Singular Rent Model

3.1 Introduction

The classical rent theory reformulated by Sraffa (1960) has been widely developed and generalised (for instance, see Montani 1975, Kurz 1978, Quadrio-Curzio 1980, D'Agata 1983, Salvadori 1986, Kurz and Salvadori 1995, Chap. 10, Bidard 2010, 2014). Based on different assumptions, three kinds of rent are usually discussed: extensive rent, intensive rent and external differential rent. In much of the literature, requirements for use are explicitly or implicitly assumed to be given. The only exception is the model built by Salvadori (1983), who discovered a new variety of rent called singular rent or demand-led rent. In his model, requirements for use are functions of income distribution, and rent can arise and is determined by the conditions of production as well as the conditions of requirements for use.

In order to separate singular rent from other kinds of rent, Salvadori (1983) only considers a single system of production, and no choice of technique is discussed. However, as pointed out by Kurz and Salvadori, "With land there is always a problem of the choice of technique to be solved (Kurz and Salvadori 1995, p.307)". The purpose of this chapter is to make a simple generalisation to the singular rent model built by Salvadori (1983) by introducing many agricultural processes. As a consequence, the problem of choice of technique arises. The cost of this simple generalisation is, however, mixing singular rent with other kinds of rents. More specifically, it is impossible to exclude intensive differential rent from the model that will be discussed below.[1] Due to the complexities of the problem, no general condition for the existence of the cost-minimising technique will be provided. This chapter only seeks to show how the choice of technique can be tackled by using a graphical tool.

Some results obtained by this chapter can be summarised as follows. If there are many agricultural processes which can be chosen, then according to the criterion of cost-minimising, some unpleasant situations similar to joint production discussed by Salvadori (1982) can exist if only singular rent is considered. These situations are: more than one cost-minimising technique may exist even though joint production is set aside, and there may be no cost-minimising technique even though all techniques are feasible. However, in the latter case, it can be shown that the system determining intensive rent is feasible and cost-minimising.

This chapter is structured as follows. Section 3.2 will make some basic assumptions and provide the basic framework for later sections. A numerical example is given in section 3.3 to show what problems we may encounter if choice of technique is considered. Section 3.4 will discuss the problem of choice of technique more formally. Some conclusions will be given in section 3.5.

3.2 Basic Assumptions and the Framework

The model is based on the following assumptions. Assume that there are n commodities which can be produced by m ($m \geqq n$) constant-returns-to-scale processes in the economy. The commodities are divided into two

[1]Since the analysis of external differential rent is similar to that of intensive rent, the results obtained in this chapter also apply to the situation that there exist many industrial processes. In such a situation, the model will mix singular rent with external differential rent.

groups: agricultural commodities, whose production uses land directly, and industrial commodities, whose production does not use land directly. To simplify the analysis, it is assumed that only commodity n (corn) is an agricultural commodity, and the others are industrial commodities. The following assumptions hold.

Assumption 3.1. *There exists only one quality of land, whose quantity is given as t units.*

Assumption 3.2. *There exists one process for the production of each industrial commodity.*

Assumption 3.3. *Each process produces one and only one commodity.*

From assumption 3.2 we know that there are $m - n + 1$ processes which can produce corn. Reorder all the processes such that the last $m - n + 1$ processes produce corn. Each process i is represented by (a_i, c_i, l_i, b_i), where a_i is a semi-positive commodity input vector, c_i is a land input scalar, l_i is a labour input scalar, b_i is a semi-positive output vector. For process j, if $1 \leq j \leq n-1$, $c_j = 0$. Since joint production is set aside, we can normalise all the processes such that each process produces 1 unit of commodity, i.e., for each process i, $b_i = e_i$ if $1 \leqq i \leqq n-1$, and $b_i = e_n$ if $n \leqq i \leqq m$, where e_i is the ith unit vector, $i = 1, 2, \cdots, n$.

We will use the indirect approach (the term used by Kurz and Salvadori 1995, Chap. 5) to discuss the problem of choice of technique. A technique is defined as a set of n processes and each of them produces a different commodity i $(i = 1, 2, \cdots, n)$. A technique is represented by $(A^{(h)}, c^{(h)}, l^{(h)}, I)$, where I is an identity matrix, and

$$A^{(h)} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^{(h)T} \end{bmatrix}_{n\times n} \qquad c^{(h)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_n^{(h)} \end{pmatrix}_{n\times 1} \qquad l^{(h)} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n^{(h)} \end{pmatrix}_{n\times 1}$$

The process $(a_n^{(h)}, c_n^{(h)}, l_n^{(h)}, e_n)$ is one of the $m-n+1$ processes producing corn. The number of all techniques is $v \equiv m - n + 1$.

In order to simplify the analysis, we assume that all commodities are basic commodities, i.e., each commodity j enters directly or indirectly into the production of all commodities, and that each technology is viable. More formally, we assume the following assumptions hold.

Assumption 3.4. *The matrix $A^{(h)}$ is indecomposable, $h = 1, 2, \cdots, v$.*

Assumption 3.5. *For each $A^{(h)}$, there exists a semi-positive vector x such that $x^T \geqq x^T A^{(h)}$.*

From the above two assumptions we know that $\lambda^F(A^{(h)}) < 1$, where $\lambda^F(A^{(h)})$ is the Perron-Frobenius root of $A^{(h)}$.

Society is divided into three groups of people: workers, who provide labour and receive wages, landlords, who own land and receive rents by renting land to capitalists, and capitalists, who hire labour, rent land from landlords, make investments and gain profits. As regards consumption patterns, it is assumed that workers, capitalists and landlords consume different commodities in different proportions. More specifically, we use b_w, b_c, and b_l to denote the consumption baskets of workers, capitalists and landlords, respectively.

Let x be the intensity vector, p the price vector, w the wage rate, r the rate of profit, and q the rent rate (per unit of land). A technique $(A^{(h)}, c^{(h)}, l^{(h)}, I)$ is feasible for a given rate of profit r if there exist x, p, w, q such that the following system holds.

$$p = (1+r)A^{(h)}p + wl^{(h)} + qc^{(h)} \tag{3.1a}$$

$$x^T = x^T A^{(h)} + w\frac{x^T l^{(h)}}{b_w^T p} b_w^T + r\frac{x^T A^{(h)} p}{b_c^T p} b_c^T + q\frac{x^T c^{(h)}}{b_l^T p} b_l^T \tag{3.1b}$$

$$x^T c^{(h)} \leq t \tag{3.1c}$$

$$x^T c^{(h)} q = tq \tag{3.1d}$$

$$u^T p = 1 \tag{3.1e}$$

$$x \geqq 0, \quad p \geqq 0, \quad w \geq 0, \quad q \geq 0 \tag{3.1f}$$

In the above system, u is a semi-positive vector which is used as numéraire. It is well known (for instance, see Kurz and Salvadori 1995, Chap. 4.)

that if $r > R^{(h)}$, where $R^{(h)} = (1 - \lambda(A^{(h)}))/\lambda(A^{(h)})$ is the maximum rate of profit that technique $(A^{(h)}, c^{(h)}, l^{(h)}, I)$ is able to pay, then technique $(A^{(h)}, c^{(h)}, l^{(h)}, I)$ is not feasible.

From equation (3.1a) and (3.1e) we know that for a single technique:

$$w = \frac{1}{u^T[I-(1+r)A^{(h)}]^{-1}l^{(h)}} - q\frac{u^T[I-(1+r)A^{(h)}]^{-1}c^{(h)}}{u^T[I-(1+r)A^{(h)}]^{-1}l^{(h)}} \tag{3.2a}$$

$$p = w[I-(1+r)A^{(h)}]^{-1}l^{(h)} + q[I-(1+r)A^{(h)}]^{-1}c^{(h)} \tag{3.2b}$$

Let the total amount of employment be L. If choice of technique is not considered, for instance, assume that there only exists technique 1, and normalise b_w, b_c and b_l as follows:[2]

$$b_w^T(I-A^{(1)})^{-1}l^{(1)} = b_c^T(I-A^{(1)})^{-1}l^{(1)} = b_l^T(I-A^{(1)})^{-1}l^{(1)} = 1 \tag{3.3}$$

Salvadori (1983) (see also Kurz and Salvadori 1995, Chap. 10) proved that:

Case (i): $x^T c^{(h)} < t$, or land is not scarce. Then $q = 0$ and p, w and x are determined in the same way as the single production technique without using land (for instance, see Kurz and Salvadori 1995, Chap. 4).

Case (ii): $x^T c^{(h)} = t$, or land is scarce. Then q, p and w are jointly determined by system (3.2) and the following equation (see Salvadori 1983, p. 84, equation (9), or Kurz and Salvadori 1995, p. 302, equation (10.17)):

$$\frac{t}{L} = b_c^T(I-A^{(1)})^{-1}c^{(1)} + w\frac{(b_w-b_c)^T(I-A^{(1)})^{-1}c^{(1)}}{b_w^T p} + \frac{qt}{L}\frac{(b_l-b_c)^T(I-A^{(1)})^{-1}c^{(1)}}{b_l^T p} \tag{3.4}$$

The above two cases can be shown graphically. Note that equation (3.2a) represents a straight line in the (q, w) plane, and from system (3.2) and (3.4) we can get a second degree algebraic equation of q:

$$\alpha q^2 + \beta q + \gamma = 0 \tag{3.5}$$

[2] It should be noted that this normalisation, which simplifies the exposition, has no influence on the solutions of the system.

where α, β, and γ are determined by the condition of production technique and rate of profit r. Define $F(q) = \alpha q^2 + \beta q + \gamma$, and we can draw the function $F(q)$ and equation (3.2a) in two related planes (see Figure 3.1).

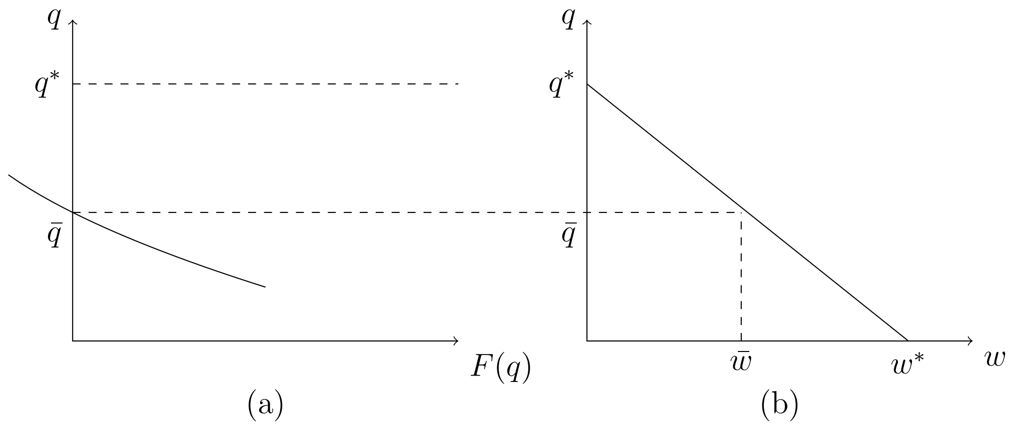

Figure 3.1: w-q relationship for a single technique

In Figure 3.1, if land is not scarce, then rent is zero and the wage rate is w^*. If land is scarce, then rent is determined by equation 3.5, or the intersection(s) of $F(q)$ with the vertical axis in Figure 3.1(a), and the wage rate is determined by equation (3.2a). Since $F(q)$ is a second degree algebraic equation of q, it may have two, or one or no intersection with the vertical axis within the area $[0, q^*]$. Therefore there may exist two, or one, or no solution when land is scarce.

Now assume that there exist many techniques, and we introduce the criterion of cost-minimising. A feasible technique $(A^{(h)}, c^{(h)}, l^{(h)}, I)$ is called a cost-minimising technique at rate of profit r if, at price vector p, wage rate w and rent q corresponding to r, no process is able to pay extra profits. In other words, for a given rate of profit r, the vector $x^{(h)}$, $p^{(h)}$ and scalars $w^{(h)}$ and $q^{(h)}$ is a solution to system (3.1), and if $(A^{(h)}, c^{(h)}, l^{(h)}, I)$ is cost-minimising, then for all techniques, we have:

$$p^{(h)} \leqq (1+r)A^{(k)}p^{(h)} + w^{(h)}l^{(k)} + q^{(h)}c^{(k)} \quad k = 1, 2, \cdots v. \tag{3.6}$$

and the equation holds when $k = h$.

Before embarking on the main discussion, an example is given first to show what problems may occur when choice of technique is considered.

3.3 An Example

We will consider a simple example as follows. There are three processes producing two commodities: corn, which is an agricultural commodity, and iron, which is an industrial commodity. The input-output pattern is shown in table 3.1. The quantity of land is 10 units, total employment of labour is 45 units. Assume that workers consume only corn, and capitalists and landlords consume only iron. Corn is used as numéraire.

We define processes (1) and (2) as technique 1, and processes (1) and (3) as technique 2. It can be checked that $R^{(1)} = \frac{41-3\sqrt{145}}{4}$ and $R^{(2)} = \frac{27-2\sqrt{106}}{5}$. Therefore if $r > \frac{27-2\sqrt{106}}{5}$, no technique is feasible. We assume that rate of profit r belongs to $[0, \frac{27-2\sqrt{106}}{5}]$.

Denote the price of iron as p_1. For technique 1, we have the following system:

$$\frac{3}{2}w + (\frac{p_1}{3} + \frac{1}{10})(r+1) = p_1 \tag{3.7a}$$

$$q + w + (\frac{p_1}{3} + \frac{1}{6})(r+1) = 1 \tag{3.7b}$$

$$\tag{3.7c}$$

or:

$$w = \frac{30rq - 60q - 41r + 2r^2 + 47}{15(r+7)} \tag{3.8}$$

$$p_1 = \frac{9(9 - r - 10q)}{10(r+7)} \tag{3.9}$$

Let x_1 and x_2 be the intensities of the processes. The following system holds:

Table 3.1: Input-Output Patterns

	Inputs					Outputs	
	Iron	Corn	Labour	Land		Iron	Corn
Iron	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{3}{2}$	0	$\rightarrow$	1	0
Corn	$\frac{1}{3}$	$\frac{1}{6}$	1	1	$\rightarrow$	0	1
Corn	$\frac{1}{4}$	$\frac{1}{5}$	1	2	$\rightarrow$	0	1

$$\frac{3}{2}x_1 + x_2 = 45 \tag{3.10a}$$

$$q(10 - x_2) = 0 \tag{3.10b}$$

$$x_1 = \frac{1}{3}x_1 + \frac{1}{3}x_2 + [x_1(\frac{p_1}{3} + \frac{1}{10}) + x_2(\frac{p_1}{3} + \frac{1}{6})]\frac{r}{p_1} + \frac{10q}{p_1} \tag{3.10c}$$

$$x_2 = \frac{1}{10}x_1 + \frac{1}{6}x_2 + 45w \tag{3.10d}$$

It can be checked that:

(1) If land is not scarce:

$$p_1 = \frac{9(9-r)}{10(r+7)} \tag{3.11}$$

$$w = -\frac{2r^2 - 41r + 47}{15(r+7)} \tag{3.12}$$

$$q = 0 \tag{3.13}$$

$$x_1 = \frac{10(-4r^2 + 107r + 81)}{9(r+7)} \tag{3.14}$$

$$x_2 = \frac{20(r^2 - 20r + 27)}{3(r+7)} \tag{3.15}$$

The above solution can only exist if $r \in [0, \frac{41-3\sqrt{145}}{4}]$.

(2) If land is scarce:

$$p_1 = \frac{3(r+3)}{10(2-r)} \tag{3.16}$$

$$w = \frac{2}{15} \tag{3.17}$$

$$q = \frac{2r^2 - 43r + 33}{30(2-r)} \tag{3.18}$$

$$x_1 = \frac{70}{3} \tag{3.19}$$

$$x_2 = 10 \tag{3.20}$$

The above solution can only exist if $r \in [0, \frac{43-\sqrt{1585}}{4}]$.

For technique 2, we have:

$$\frac{3}{2}w + (\frac{p_1}{3} + \frac{1}{10})(r+1) = p_1 \tag{3.21a}$$

$$2q + w + (\frac{p_1}{4} + \frac{1}{5})(r+1) = 1 \tag{3.21b}$$

or:

$$w = \frac{80rq - 160q - 54r + 5r^2 + 61}{5(r+25)} \tag{3.22}$$

$$p_1 = \frac{-12(2r + 30q - 13)}{5(r+25)} \tag{3.23}$$

The following system holds:

$$\frac{3}{2}x_1 + x_2 = 45 \tag{3.24a}$$

$$q(10 - 2x_2) = 0 \tag{3.24b}$$

$$x_1 = \frac{1}{3}x_1 + \frac{1}{4}x_2 + [x_1(\frac{p_1}{3} + \frac{1}{10}) + x_2(\frac{p_1}{4} + \frac{1}{5})]\frac{r}{p_1} + \frac{10q}{p_1} \tag{3.24c}$$

$$x_2 = \frac{1}{10}x_1 + \frac{1}{5}x_2 + 45w \tag{3.24d}$$

We have: (1) if land is not scarce:

$$p_1 = \frac{12(13 - 2r)}{5(r + 25)} \tag{3.25}$$

$$w = \frac{5r^2 - 54r + 61}{5(r + 25)} \tag{3.26}$$

$$q = 0 \tag{3.27}$$

$$x_1 = \frac{90(-5r^2 + 58r + 39)}{13(r + 25)} \tag{3.28}$$

$$x_2 = \frac{45(15r^2 - 161r + 208)}{13(r + 25)} \tag{3.29}$$

The above solution can only exist if $r \in [0, \frac{27-2\sqrt{106}}{5}]$.

(2) if land is scarce:

$$p_1 = \frac{9r + 13}{30(2 - r)} \tag{3.30}$$

$$w = \frac{4}{135} \tag{3.31}$$

$$q = \frac{135r^2 - 1462r + 1547}{2160(2 - r)} \tag{3.32}$$

$$x_1 = \frac{80}{3} \tag{3.33}$$

$$x_2 = 5 \tag{3.34}$$

The above solution can only exist if $r \in [0, \frac{731-2\sqrt{81379}}{135}]$.

Therefore, if $r \in [0, \frac{43-\sqrt{1585}}{4}]$, both technique 1 and technique 2 are feasible no matter whether or not land is scarce. However, if $r \in (90 - \sqrt{7969}, \frac{43-\sqrt{1585}}{4})$ and land is scarce, it can be checked that the following inequalities hold:

$$p^{(1)} \geq (1+r)A^{(2)}p^{(1)} + w^{(1)}l^{(2)} + q^{(1)}c^{(2)} \tag{3.35a}$$
$$p^{(2)} \geq (1+r)A^{(1)}p^{(2)} + w^{(2)}l^{(1)} + q^{(2)}c^{(1)} \tag{3.35b}$$

In other words, neither technique 1 nor technique 2 is cost-minimising. In what follows we ignore the situation when technique 1 and technique 2 are equally profitable. It can be checked that if $r \in [0, 90-\sqrt{7969})$ and land is not scarce, then technique 2 is cost-minimising; if r belongs to the same region and land is scarce, then technique 1 is cost-minimising. If $r \in (90-\sqrt{7969}, \frac{43-\sqrt{1585}}{4})$ and land is not scarce, then technique 2 is cost-minimising; if r belongs to the same region and land is scarce, both techniques are feasible but neither is cost-minimising. If $r \in (\frac{43-\sqrt{1585}}{4}, \frac{761-38\sqrt{391}}{9})$ and land is not scarce, then technique 2 is cost-minimising; if r belongs to the same region and land is scarce, only technique 2 is feasible but it is not cost-minimising. If $r \in (\frac{761-38\sqrt{391}}{9}, \frac{731-2\sqrt{81379}}{135})$ and land is not scarce, then technique 2 is cost-minimising; if r belongs to the same region and land is scarce, only technique 2 is feasible but it is cost-minimising. If $r \in (\frac{731-2\sqrt{81379}}{135}, \frac{41-3\sqrt{145}}{4})$ and land is not scarce, technique 2 is cost-minimising; if r belongs to the same region, no technique is feasible. If $r \in (\frac{41-3\sqrt{145}}{4}, \frac{27-2\sqrt{106}}{5})$ and land is not scarce, then only technique 2 is feasible and it is cost-minimising; if r belongs to the same region and land is scarce, then no technique is feasible.

Up to now we have not considered intensive rent. Since there is more than one process which can produce corn, intensive rent can also exist. Therefore if land is scarce, we can also have the following system:

$$\frac{3}{2}w + (\frac{p_1}{3} + \frac{1}{10})(r+1) = p_1 \tag{3.36a}$$
$$q + w + (\frac{p_1}{3} + \frac{1}{6})(r+1) = 1 \tag{3.36b}$$
$$2q + w + (\frac{p_1}{4} + \frac{1}{5})(r+1) = 1 \tag{3.36c}$$

We have:

$$p_1 = \frac{12(14-r)}{5(7r+31)} \tag{3.37}$$

$$w = \frac{r^2 - 166r + 193}{15(7r+31)} \tag{3.38}$$

$$q = \frac{-13r^2 + 40r + 53}{30(7r+31)} \tag{3.39}$$

Let the intensities be x_1, x_2 and x_3. We then have the following system:

$$\frac{3}{2}x_1 + x_2 + x_3 = 45 \tag{3.40a}$$

$$q(10 - x_2 - 2x_3) = 0 \tag{3.40b}$$

$$x_1 = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 + [x_1(\frac{1}{3}p_1 + \frac{1}{10}) + x_2(\frac{1}{3}p_1 + \frac{1}{6}) + x_3(\frac{1}{4}p_1 + \frac{1}{5})]\frac{r}{p_1} + \frac{10q}{p_1} \tag{3.40c}$$

$$x_2 + x_3 = \frac{1}{10}x_1 + \frac{1}{6}x_2 + \frac{1}{5}x_3 + 45w \tag{3.40d}$$

We have:

$$x_1 = \frac{5(-9r^2 + 2306r + 1859)}{21(7r+31)} \tag{3.41}$$

$$x_2 = \frac{5(9r^2 - 1552r + 1613)}{7(7r+31)} \tag{3.42}$$

$$x_3 = \frac{-45(r^2 - 180r + 131)}{14(7r+31)} \tag{3.43}$$

The above solutions exist if $r \in [90 - \sqrt{7969}, \frac{761-38\sqrt{391}}{9}]$. From the above analysis we know that if $r \in (90 - \sqrt{7969}, \frac{43-\sqrt{1585}}{4})$, both technique 1 and 2 are feasible, but neither of them is cost-minimising. However, in this region, the system determining intensive rent is feasible and it is cost-minimising.

It is possible to provide other examples with more commodities and examples that capitalists and landlords consume different commodities in different proportions. However, those examples, although may be more realistic, only generate more complex numbers and equations, but will not yield more results than this simple one.

From this simple example we can draw some preliminary conclusions with respect to the problem of choice of technique. First, there may exist more than one cost-minimising technique even though joint production is set aside. Second, the existence of feasible technique does not guarantee the existence of cost-minimising technique. Third, if land is scarce and all techniques are feasible but no cost-minimising technique with regard to singular rent exists, then the system determining intensive rent may be feasible and may be the only cost-minimising system. We will prove this in the next section.

3.4 Choice of Technique

Based on the framework in section 3.2 and the example in section 3.3, we discuss the problem of choice of technique more formally in this section. Due to the complexities of the problem, no general condition for the existence of cost-minimising technique will be provided, and we will only show how choice of technique can be dealt with graphically.

This section proceeds as follows. Since intensive rent is determined in a different way from singular rent, first we deal with the problem of choice of technique with respect to singular rent alone, and consider intensive rent later. Different cases are classified and discussed separately. It shows that more than one cost-minimising technique may exist, and that there may be no cost-minimising technique even though all techniques are feasible. Intensive rent is then discussed, and we will show that in the latter case, when no cost-minimising technique exists but all techniques are feasible, the intensive rent arises and the system determining intensive rent is the only cost-minimising system.

3.4.1 Choice of technique with respect to singular rent

We use the graphical tool to discuss the problem of choice of technique. For each technique $(A^{(k)}, c^{(k)}, l^{(k)}, I)$, $k = 1, 2, \cdots, v$, we can draw straight lines (q, w) corresponding to equation (3.2a) and the curves corresponding to the second algebraic degree function $F(q)$.[3]

For the purpose of illustration, we only consider an example with two techniques (technique 1 and 2), and rewrite equations (3.2a) and (3.5) as follows.

For technique 1:

$$w = \frac{1}{u^T[I-(1+r)A^{(1)}]^{-1}l^{(1)}} - q\frac{u^T[I-(1+r)A^{(1)}]^{-1}c^{(1)}}{u^T[I-(1+r)A^{(1)}]^{-1}l^{(1)}} \tag{3.44a}$$

$$\alpha_1 q^2 + \beta_1 q + \gamma_1 = 0 \tag{3.44b}$$

For technique 2:

$$w = \frac{1}{u^T[I-(1+r)A^{(2)}]^{-1}l^{(2)}} - q\frac{u^T[I-(1+r)A^{(2)}]^{-1}c^{(2)}}{u^T[I-(1+r)A^{(2)}]^{-1}l^{(2)}} \tag{3.45a}$$

$$\alpha_2 q^2 + \beta_2 q + \gamma_2 = 0 \tag{3.45b}$$

Only the straight lines corresponding to these two techniques are drawn (Figure 3.2). Remember that these lines (and curves) change when the rate of profit r changes. Since there exists only one process in technique 1 that is different from those in technique 2, $[I-(1+r)A^{(1)}]^{-1}l^{(1)}$ is either strictly larger than, or equal to, or strictly less than $[I-(1+r)A^{(2)}]^{-1}l^{(2)}$,[4] we suppose

[3]For a technique $(A^{(k)}, c^{(k)}, l^{(k)}, I)$, $k \neq 1$, the structure of equation (3.4) is more complex, because of the fact that equation (3.3) is not satisfied, but equation (3.5) is still a second degree algebraic form.

[4]This can be shown briefly as follows. If for a given rate of profit r, land is not scarce in both techniques, and let the price and the wage rate for technique i be p_i and w_i $(i = 1, 2)$, respectively. Then technique 2 either pays extra profits, or is equally profitable, or incurs extra costs at p_1 and w_1 for the given rate of profit r (because there exists only one process in technique 1 that is different from those in technique 2). We assume, for instance, technique 2 incurs extra costs at p_1 and w_1, or:

$$p_1 \leq (1+r)A_2 p_1 + w_1 l_2$$

the following inequality holds:

$$[I - (1+r)A^{(1)}]^{-1}l^{(1)} < [I - (1+r)A^{(2)}]^{-1}l^{(2)} \tag{3.46}$$

The above inequality is equivalent to the following one:

$$u^T[I - (1+r)A^{(1)}]^{-1}l^{(1)} < u^T[I - (1+r)A^{(2)}]^{-1}l^{(2)} \tag{3.47}$$

The above inequality means that if rents are zero, technique 1 pays a higher wage rate than technique 2 at the given rate of profit r.

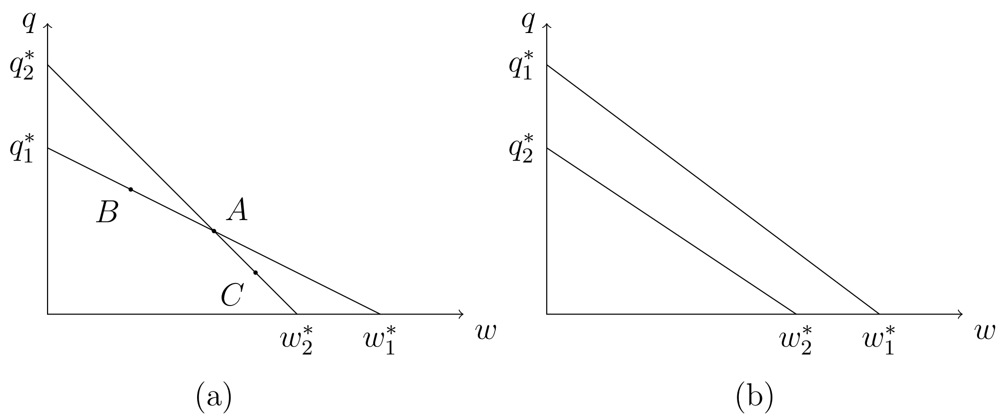

Figure 3.2: An example of choice of technique

Now several cases will be discussed.

Case (i): Land is not scarce in both techniques at the given rate of profit.

In this case, the cost-minimising technique is the one which pays the highest wage rate.[5] In Figure 3.2, the cost-minimising technique is technique

Or we have:

$$\frac{p_1}{w_2}[I - (1+r)A_2] \leq l_2$$

The matrix $[I - (1+r)A_2]$ is invertible and the inverse is positive because matrix A is indecomposable and $r < R^{(2)}$, then we have:

$$[I - (1+r)A_1]^{-1}l_1 = \frac{p_1}{w_1} < [I - (1+r)A_2]^{-1}l_2$$

[5]See Kurz and Salvadori 1995, Chap. 5

1.

Case (ii) Land is scarce if technique 1 is used, but it is not scarce if technique 2 is used.

In this case, technique 2 cannot be cost-minimising, because at $w = w_2^*$, technique 1 always pays extra profits. The only possible cost-minimising technique (but not necessarily existing) is technique 1. We need to consider the relationship between these two straight lines. If the following inequality holds:

$$u^T[I - (1+r)A^{(1)}]^{-1}c^{(1)} < u^T[I - (1+r)A^{(2)}]^{-1}c^{(2)}$$

then the two straight lines do not intersect (since we already assume that inequality (3.47) holds, see Figure 3.2b). In this case, the cost-minimising technique can only be technique 1, because at any possible rent-wage pair determined by technique 1, technique 2 incurs extra costs. Hence the cost-minimising technique exists if the second degree function (3.44b) has solution(s) within $[0, q_1^*]$.

If the following inequality holds:

$$u^T[I - (1+r)A^{(1)}]^{-1}c^{(1)} > u^T[I - (1+r)A^{(2)}]^{-1}c^{(2)}$$

then the two straight lines intersect (Figure 3.2a), and the cost-minimising technique exists if the solution(s) $(\bar{q}_1, \bar{w}_1)$ of system (3.44) lies within the line Aw_1^*. Conversely, if the condition does not hold, and suppose that the solution $(\bar{q}_1, \bar{w}_1)$ of system (3.44) is point B, then we see that, at $(\bar{q}_1, \bar{w}_1)$, technique 2 pays extra profits and, at $(0, w_2^*)$, technique 1 pays extra profits, and no cost-minimising technique exists.

Case (iii): Land is scarce in both techniques.

We still assume that $u^T[I-(1+r)A^{(1)}]^{-1}c^{(1)} > u^T[I-(1+r)A^{(2)}]^{-1}c^{(2)}$ and the two straight lines have an intersection. In this case, the cost-minimising technique exists if the solution(s) of system (3.44) lies within the line Aw_1^*, or if the solution(s) of system (3.45) lies within the line Aq_2^*. If this condition holds, then there may exist more than one cost-minimising technique.

Suppose the opposite, the solution of system (3.44) is point B, and the solution of system (3.45) is point C. Then at (q_B, w_B), technique 2 pays extra profits, and at (q_C, w_C), technique 1 pays extra profits, and there exists no cost-minimising technique, even though all techniques are feasible.

3.4.2 Intensive rent

In the above analysis, we have not considered intensive rent. When corn is produced by one agricultural process alone and land becomes scarce, instead of replacing this process with another agricultural process, these two processes can be operated together, provided that the second process uses less land but is more expensive. Intensive rent arises in such a situation, and is determined by the condition of production alone.

Let us continue to consider the above two techniques. Let $(\bar{A}, \bar{c}, \bar{l}, \bar{B})$ represent all the processes in $(A^{(1)}, c^{(1)}, l^{(1)}, I)$ and the last process in $(A^{(2)}, c^{(2)}, l^{(2)}, I)$. For instance, in the example presented in section 3.3, $(\bar{A}, \bar{c}, \bar{l}, \bar{B})$ is composed by technique 1 (process (1) and process (2)) and the last process in technique 2 (process (3)). Let $\bar{q}$, $\bar{w}$, $\bar{p}$ be the intensive rent and the corresponding wage rate and prices, respectively. We have:

$$\bar{B}\bar{p} = (1+r)\bar{A}\bar{p} + \bar{w}\bar{l} + \bar{q}\bar{c} \tag{3.48a}$$
$$u^T\bar{p} = 1 \tag{3.48b}$$
$$\bar{p} \geqq 0, \quad \bar{w} \geq 0, \quad \bar{q} \geq 0 \tag{3.48c}$$

Intensive rent can also be shown using the above graph. System (3.48) is equivalent to the following system:

$$\bar{p} = (1+r)A^{(1)}\bar{p} + \bar{w}l^{(1)} + \bar{q}c^{(1)} \tag{3.49a}$$
$$\bar{p} = (1+r)A^{(2)}\bar{p} + \bar{w}l^{(2)} + \bar{q}c^{(2)} \tag{3.49b}$$
$$u^T\bar{p} = 1 \tag{3.49c}$$
$$\bar{p} \geqq 0, \quad \bar{w} \geq 0, \quad \bar{q} \geq 0 \tag{3.49d}$$

or:

$$\bar{w} = \frac{1}{u^T[I-(1+r)A^{(1)}]^{-1}l^{(1)}} - \bar{q}\frac{u^T[I-(1+r)A^{(1)}]^{-1}c^{(1)}}{u^T[I-(1+r)A^{(1)}]^{-1}l^{(1)}} \tag{3.50a}$$

$$\bar{w} = \frac{1}{u^T[I-(1+r)A^{(2)}]^{-1}l^{(2)}} - \bar{q}\frac{u^T[I-(1+r)A^{(2)}]^{-1}c^{(2)}}{u^T[I-(1+r)A^{(2)}]^{-1}l^{(2)}} \tag{3.50b}$$

$$\bar{w} \geq 0, \quad \bar{q} \geq 0 \tag{3.50c}$$

The above system represents two straight lines in plane (w, q), which have the same structures as equation (3.44a) and equation (3.45a). Hence intensive rent exists only when the above two straight lines have an intersection in the semipositive quadrant. In Figure 3.2, intensive rent does not exist in Figure 3.2b; it only exists in Figure 3.2a. The scale of intensive rent is determined by the intersection of the two straight lines, that is point A in Figure 3.2a. Therefore intensive rent is determined differently from singular rent.

As regards the quantity system, we use z to denote the intensity vector, and we have:

$$z^T\bar{B} = z^T\bar{A} + \bar{w}\frac{z^T\bar{l}}{b_w^T\bar{p}}b_w^T + r\frac{z^T\bar{A}\bar{p}}{b_c^T\bar{p}}b_c^T + \bar{q}\frac{z^T\bar{c}}{b_l^T\bar{p}}b_l^T \tag{3.51a}$$

$$z^T\bar{c} = t \tag{3.51b}$$

$$z \geqq 0 \tag{3.51c}$$

System (3.51) has $n+1$ equations with $n+1$ knowns. Since equation (3.48a) holds, one of the equations of (3.51a) is linearly dependent on the others, and system (3.51) has one degree of freedom. Hence system (3.51) always has a non-trivial solution z.

Let $\bar{z}_n \equiv \frac{t}{\bar{c}_n}$ and $\bar{z}_{n+1} \equiv \frac{t}{\bar{c}_{n+1}}$ be the maximum intensities of the two agricultural processes, respectively. It can be easily checked that for any $\lambda \in [0,1]$, $\lambda\bar{z}_n$ and $(1-\lambda)\bar{z}_{n+1}$ satisfy equation (3.51b). Since there exists one degree of freedom in system (3.51), if we set one of the two agricultural process intensities as exogenous, then the system is fully determined, and it can be shown that the system has a semipositive solution. Without loss of generality, let the intensity of the last process be $z^*_{n+1} \in [0, \bar{z}_{n+1}]$. We can then prove the following proposition:

Proposition 3.1. *If* $z^*_{n+1} \in [0, \bar{z}_{n+1}]$ *is given, then system* (3.51) *has a semipositive solution.*

The proof of the proposition is based on the following lemma.

Lemma 3.1. *Taken* $\bar{p}$, $\bar{w}$ *and* $\bar{q}$ *determined by system* (3.48) *as given, the following two systems have positive solutions:*

$$y^T = y^T A^{(1)} + \bar{w}\frac{y^T l^{(1)}}{b_w^T \bar{p}} b_w^T + r\frac{y^T A^{(1)}\bar{p}}{b_c^T \bar{p}} b_c^T + \bar{q}\frac{y^T c^{(1)}}{b_l^T \bar{p}} b_l^T \tag{3.52a}$$

$$y^T c^{(1)} = t \tag{3.52b}$$

$$y \geqq 0 \tag{3.52c}$$

$$y^T = y^T A^{(2)} + \bar{w}\frac{y^T l^{(2)}}{b_w^T \bar{p}} b_w^T + r\frac{y^T A^{(2)}\bar{p}}{b_c^T \bar{p}} b_c^T + \bar{q}\frac{y^T c^{(2)}}{b_l^T \bar{p}} b_l^T \tag{3.53a}$$

$$y^T c^{(2)} = t \tag{3.53b}$$

$$y \geqq 0 \tag{3.53c}$$

Proof of Lemma 3.1.

Without loss of generality, we will prove there exists a positive solution to system (3.52).

Define matrix M as follows:

$$M = [A^{(1)} + \frac{\bar{w}}{b_w^T \bar{p}} l^{(1)} b_w^T + \frac{r}{b_c^T \bar{p}} A^{(1)} \bar{p} b_c^T]$$

Since equation (3.52b) holds, equation (3.52a) becomes as follows:

$$y^T [I - M] = \frac{\bar{q}t}{b_l^T \bar{p}} b_l^T \tag{3.54}$$

Since equation (3.49a) holds, we have:

$$[I - M]\bar{p} = \bar{q}c^{(1)}$$

i.e., there exists a semipositive vector $\bar{p}$ such that $[I - M]\bar{p}$ is semipositive. Further M is indecomposable because $A^{(1)}$ is indecomposable. Therefore, $[I-M]$ is invertible and the inverse is positive (Nikaido 1968, p. 107, Theorem 7.4). Multiply the inverse of $[I - M]$ on both sides of equation (3.54) and we get the solution y^* of system (3.52). Similarly, there exists a solution y' to system (3.53).

Q.E.D.

Now we can prove the proposition.

Proof of Proposition 3.1.

From Lemma 3.1 we know that there exist positive solutions y^* and y' to system (3.52) and (3.53), respectively. In addition, we have $y_n^* = \bar{z}_n$ and $y'_n = \bar{z}_{n+1}$, hence $z_{n+1}^* \leqq y'_n$. Let $z_{n+1}^* = \lambda y'_n$, where $0 \leq \lambda \leq 1$. Then z^* is a solution to system (3.51), where:

$$\begin{aligned} z_i^* = \lambda y_i^* + (1-\lambda)y'_i \quad \text{for } i = 1 \cdots n-1 \\ z_n^* = \lambda y_n^* \\ z_{n+1}^* = (1-\lambda)y'_n \end{aligned}$$

Q.E.D.

Now we come back to the above cases discussed in section 3.4.1. In the last case we showed that the cost-minimising technique may not exist even though all the techniques are feasible. However, such a situation can only happen when the two straight lines in Figure 3.2a have an intersection, and this is the condition when intensive rent exists. From Figure 3.2a we can also see that no technique is able to pay extra profits at $(\bar{p}, \bar{w}, \bar{q})$. Therefore the system determining intensive rent is cost-minimising. Taking one of the agricultural intensities as given, the quantity system can be fully determined.

3.5 Concluding Remarks

If requirements for use are functions of income distribution, and social classes consume commodities in different proportions, then a new kind of rent, that is singular rent or demand-led rent, can arise when land is scarce. As shown by Salvadori (1983) (see also Kurz and Salvadori 1995, Chap. 10), singular rent is determined by the conditions of production as well as the conditions of requirements for use.

This chapter made a simple generalisation to the singular rent model, that is, to introduce many agricultural processes and to consider the problem of choice of technique. Due to the complexities of the problem, no general condition for the existence of the cost-minimising technique is given, but by using a graphical tool, this chapter showed how the problem of choice of technique can be tackled.

Several different cases were discussed. It was shown that if there are many techniques that can be chosen, then with respect only to singular rent, there may exist more than one cost-minimising technique even though joint production is set aside, and that the cost-minimising technique may not exist even though all techniques are feasible. It was also shown that in the latter case the system determining intensive rent is feasible and cost-minimising.

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Chapter 4

An Exhaustible Resources Model in a Dynamic Input-Output Framework: A Possible Reconciliation Between Ricardo and Hotelling

4.1 Introduction

This chapter investigates the problem of exhaustible resources using a dynamic input-output model with classical features. Although it is controversial whether exhaustible resources will set a limit to economic growth, it is hard to deny that exhaustible resources are and will be of great importance to most economies. Solow's 1974 opinion that the topic of exhaustible (or non-renewable) resources is "important, contemporary, and perennial (Solow 1974)" is still not outdated.

The modern theory of exhaustible resources generally stems from the famous Hotelling rule (Hotelling, 1931), according to which the prices of exhaustible resources *in situ* have to increase at a rate equal to that of profit, provided free competition prevails, due to the fact that the storage of exhaustible resources requires a normal rate of profit as do other production processes. Since resources are generally used to produce energies, which are used to produce many, if not all, commodities, it is appropriate to conduct research on exhaustible resources within a multi-sectoral framework. In addition, since the Hotelling rule seems to imply that the prices of many, if not all, commodities have to change, a dynamic analysis is appropriate. Based on these two considerations, the dynamic input-output model is the prime choice to carry out the present research.

The existing literature on exhaustible resources using input-output methods usually focuses on the interdependence of exhaustible resources (or energy), economic growth and the environment, while the price evolvements of exhaustible resources are usually undiscussed or assumed to be exogenous. For instance, a rich literature is devoted to the analyses on the factors affecting energy consumption using structural decomposition analysis (SDA) in the hybrid energy input-output models (Lin and Polenske 1995, Mukhopadhyay and Chakraborty 1999, Kagawa and Inamura 2001, 2004, Dietzenbacher and Stage 2006), on the impact of non-renewable resources on economic growth (Dobos and Floriska 2005), on tracking energy paths (Treloar 1997), on reducing energy requirements (Wilting et al., 1999), and on the relations between energy and the environment (Weber and Schnabl 1998, Karten and Schleicher 1999). In these models, the prices of exhaustible resources are usually undiscussed. Dejun et al. (2013) considered the impact of prices of energy on technological coefficients, but price evolvements are exogenous. In this chapter, both economic dynamics of quantities and prices will be under comprehensive investigation in a dynamic input-output model with classical features.

It should be noted that the dynamic input-output model used in this chapter differs slightly from the dynamic Leontief model. One of the most common ways to dynamise the static input-output models is to introduce "capital" stock and the "stock-flow" matrix proposed by Leontief (1953, 1970). However, this dynamic Leontief model has caused several theoretical problems due to the rigorous assumptions made for the convenience of empirical research. To illustrate, these theoretical problems include the irreversibility of capital accumulation (Leontief 1953, McManus 1957), the singularity problem (the stock-flow matrix may be singular), causal indeterminacy (negative output may be obtained unless the economy starts from a specific composition of capital stock, see Dorfman et al., 1958, see also Takayama 1974), the stability and dual (in)stability (the quantity system is unstable, or the quantity system is stable unless the price system is unstable, and vice versa, see Sargan 1958, Morishima 1959, Solow 1959, Jorgenson 1960 and 1961, Steenge 1990). Different remedies can be made to solve these problems, for instance, Leontief himself (1953) proposed a "multi-phase" process to deal with the irreversibility; the singularity problem can also be solved through different ways (Kendrik 1972, Livesey 1973, Luenberger and Arbel 1973, Jodar and Mellero 2010.); causal indeterminacy can be avoided by allowing extra production capacity rather than full capacity utilisation (Duchin and Szyld 1985), or by changing the Leontief model into a planning model and introducing non-negativity of output (Solow 1959); and dual (in)stability can be avoided by replacing the assumption of perfect foresight with an assumption that firms make investments based on past experience (Aoki 1977). However, there still exist some imperfections in the dynamic Leontief model: first, it cannot deal with the problem of joint production, and second, even though the static input-output model has a classical tradition (Kurz and Salvadori 2000a, 2006) irrespective of some illegitimate assumptions such as given value-added coefficients in the price theory, its treatment of capital in the dynamic model is typically neo-classical which cannot survive the critiques raised by Sraffa's work (Sraffa 1960) and the following capital controversies. The second way to dynamise the static input-output model is to introduce expenditure lags, as is done by Solow (1952). However, in this kind of dynamic model the input-output matrix represents the expenditure relations rather than production technology.

Given these facts, in this chapter we use a dynamic input-output model with production lags and classical features. The model has the following advantages: first, the input-output matrix in our model still represents the production technology; second, it is possible to deal with joint production in this framework; third and more importantly, this model, which is Sraffa-von

Neumann orientated and which has a classical tradition, does not suffer from the problem of "capital" as does the neo-classical theory. Hence the present model is more logically consistent.

Before embarking on the main issue, it is appropriate to summarise the main contributions in the literature of modern classical theory on exhaustible resources, and what this chapter tries to add. The issue of exhaustible resources in the modern classical theory is in a state of intense debate. Sraffa only mentioned natural resources in passing: "Natural resources which are used in production, such as land and mineral deposits..." (Sraffa, 1960, p. 74). Several scholars subsequently investigated the theory of exhaustible resources in the Sraffian or classical framework, such as Parrinello (1983, 2001, 2004), Bidard and Erreygers (2001a, 2001b), Schefold (1989, 2001), Lager (2001), Ravagnani (2008), and Kurz and Salvadori (1995, 1997, 2000b, 2001, 2009, 2011). A survey on these contributions is produced by Kurz and Salvadori (2015). However, there is no consensus among these models, as exemplified by the symposium in Metroeconomica (2001), and the issue seems to be far from settled.

This chapter is mainly inspired by Kurz and Salvadori's contributions in this literature. Their preliminary results (Kurz and Salvadori, 1995, 1997) were revised and improved with a more elaborate version using a dynamic input-output model (Kurz and Salvadori, 2000b, 2001), which allows the time paths of prices, royalties and other endogenous variables to be tracked. In their later contribution, Kurz and Salvadori (2009) gave a new interpretation to the treatment of exhaustible resources by Ricardo, clarifying that Ricardo's analysis of exhaustible resources starts from his discussion of the difference between rent and profit in Chapter II of his *Principles* (Ricardo 1817), and that what we call "royalties" are comprised in profits in Ricardo's analysis. Based on their systematic investigation of Ricardo's work, Kurz and Salvadori point out that, although the famous Hotelling rule is not elaborated by Ricardo, it does not mean that his analysis is defective, incomplete or inferior. The differences between the analyses of Ricardo and those of Hotelling are due to different assumptions: in Ricardo's world, there are searching activities such that each exhausted mine is replaced by a newly discovered mine with the same quality and quantity, and the searching costs in terms of labour and commodities are constant. In addition, there is capacity constraint on extraction in each mine. In his theory, royalties as a sub-category of profits are not introduced explicitly, and rents of mines caused by different fertilities have nothing to do with royalties. Conversely, in Hotelling's world, the amount of one homogeneous exhaustible resource is known and given at the beginning, and the extraction of the resource at each time is only constrained by the resource remaining from the previous period. Hence, the arguments developed by Ricardo and Hotelling on exhaustible resources are derived from different assumptions, and both are helpful in improving our understanding in this issue. The ideas put forward by Ricardo and Hotelling can be incorporated into a single framework, as is done by Kurz and Salvadori (2009), who provide a formalisation of exhaustible resources with explicit capacity constraints on extraction, and who clearly distinguish three types of property incomes: profits, royalties and rents. A numerical example was subsequently given (Kurz and Salvadori, 2011) to shed more light on this issue.

This chapter, following Kurz and Salvadori's contributions (2009, 2011), seeks to make a further possible reconciliation between Ricardo's and Hotelling's analyses of exhaustible resources, that is, to introduce resource-searching activities. To be more specific, this chapter tries to contribute in the following aspects. First, it seeks to introduce searching activities and to provide a sufficient and necessary condition for the existence of solutions to the model. Given a real wage rate and a consumption vector, the chapter shows that the paths of prices, royalties, rents, intensities of commodity production and resource-searching processes can be determined, once a sequence of profit rates and the initial amounts of commodities and resources are given. Second, based on the formalisation of this chapter, we also show that some well-defined circumstances under which commodity prices are constant, discussed by Kurz and Salvadori (2009, 2011), can be represented by the present model.

Two points need to be further stressed. First, the method used in this chapter is similar to Solow's in his generalisation of the dynamic input-output system (Solow, 1959). However, the theory is totally different. Due to the difficulties faced by neo-classical capital theory, we do not assume that the distributive variables are determined by the demand for, and the supply of, factors. In this chapter, one distributive variable, the real wage rate, is assumed to be exogenously given (which means that its determination lies outside the system), and the remaining distributive variables are endogenously determined. Therefore, the model preserves some classical features.

Second, in order to make the problem manageable, perfect foresight is assumed. That is to say, when the prices are bound to change over time, the agents are aware of and know correctly how the prices change. This assumption is very strong. Since decisions made by firms and individuals are usually based on their expectations of the future, and all future states cannot be known with certainty, the discussion about expectation in principle cannot be avoided. However, introducing expectations will not only complicate the issue discussed here, but will also make it hard, if not impossible, to arrive at a certain result, because the result will depend on the assumptions made on the formation of expectations. A detailed discussion on expectations lies beyond the scope of this chapter. We just adopt one of the simplest expectations: perfect foresight, and the analysis in the following can be considered a preliminary result of a more satisfactory investigation of the subject.

The chapter is organised as follows. Basic definitions and the model are given in section 4.2. The dynamics of quantities and prices are presented in section 4.3, which gives conditions for the existence of solutions to the model. Section 4.4 discusses circumstances under which prices are constant (circumstances seem contradictory to the Hotelling rule) based on the model presented. Section 4.5 provides some conclusions.

4.2 Basic Definitions and the Model

The formalisation of the model is based on the following assumptions. Assume in the economy there are n perfectly divisible commodities, which are produced by m_1 ($m_1 > n$) constant returns to scale processes. There are s kinds of resources provided by nature, but only part of the total amount is known at the beginning, and the remainder can be discovered by m_2

($m_2 > s$) processes using commodities and labour. Instead of extracting the resources, the owners can choose to store them; hence there exist s processes for storage. Each process i producing commodities is represented by a quintuplet $(\boldsymbol{a}_i, \boldsymbol{c}_i, l_{1i}, \boldsymbol{b}_i, \mathbf{0})$, where $\boldsymbol{a}_i^T = (a_{i1}, \cdots, a_{in})$ is the commodity inputs vector,[1] $\boldsymbol{c}_i^T = (c_{i1}, \cdots, c_{is})$ is the resource inputs vector, l_{1i} is the labour input scalar, and $\boldsymbol{b}_i^T = (b_{i1}, \cdots, b_{in})$ is the commodity outputs vector. Each process j searching resources is represented by $(\boldsymbol{f}_j, \mathbf{0}, l_{2j}, \mathbf{0}, \boldsymbol{d}_j)$, where $\boldsymbol{f}_j^T = (f_{j1}, \cdots, f_{jn})$ is the commodity inputs vector, l_{2j} is the labour input scalar, and $\boldsymbol{d}_j^T = (d_{j1}, \cdots, d_{js})$ is the resource outputs vector. The whole technology at time t is represented by the following matrices:

$$\mathbf{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{m_1}]^T$$

$$\mathbf{C} = [\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_{m_1}]^T$$

$$\boldsymbol{l}_1 = [l_{11}, l_{12}, \cdots, l_{1m_1}]^T$$

$$\mathbf{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_{m_1}]^T$$

$$\mathbf{F} = [\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_{m_2}]^T$$

$$\boldsymbol{l}_2 = [l_{21}, l_{22}, \cdots, l_{2m_2}]^T$$

$$\mathbf{D} = [\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_{m_2}]^T$$

Order all the processes so that the first m_1 processes produce commodities, and the following m_2 processes search resources, and the remaining s processes store resources. All the processes are listed in table 4.1.

The technology producing commodities is assumed to be time-invariant, which means that $\mathbf{A}$, $\mathbf{C}$, $\boldsymbol{l}_1$ and $\mathbf{B}$ are constant. The technology searching resources is not necessarily time-invariant. The costs for searching one unit

[1] In this chapter, transposition of a vector or a matrix is represented by the superscript T.

Table 4.1: Input-Output Patterns

	Inputs				Outputs	
	Commodities	Resources	Labour		Commodities	Resources
Commodities	$\mathbf{A}$	$\mathbf{C}$	$\boldsymbol{l}_1$	$\rightarrow$	$\mathbf{B}$	$\mathbf{0}$
Seaching	$\mathbf{F}$	$\mathbf{0}$	$\boldsymbol{l}_2$	$\rightarrow$	$\mathbf{0}$	$\mathbf{D}$
Storage	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{0}$	$\rightarrow$	$\mathbf{0}$	$\mathbf{I}$

of a certain resource may be increasing due to the increased difficulties in locating the remaining resources. The costs for searching one unit of a certain resource can also be decreasing due to increased experience or technological progress in searching. In this chapter, in order to simplify the analysis, it is assumed that the searching costs are non-decreasing. In order to compare the searching costs among different periods, $(\mathbf{F}, \boldsymbol{l}_2)$ are assumed to be time-invariant, while the non-decreasing searching costs are represented by the changes in $\mathbf{D}$ over time. In addition, no joint searching of resources is assumed, i.e. there is only one positive component in $\boldsymbol{d}_j$, and others are zero. Formally:

Assumption 4.1. *For each $t \in \mathbb{N}$, $(\mathbf{F}, \boldsymbol{l}_2, \mathbf{0}, \mathbf{D})_t$ is known and given, and the following relations hold:*

$$(\mathbf{F}, \boldsymbol{l}_2)_t = (\mathbf{F}, \boldsymbol{l}_2)_{t+1}$$

$$\mathbf{D}_{t+1} = \alpha \mathbf{D}_t$$

where $\mathbb{N}$ is the set of all natural numbers, and $\alpha \in (0, 1]$ is a real number.

In the above assumption, if $\alpha = 1$, the searching costs in terms of commodities and labour are constant. Otherwise the searching costs are increasing.

Three types of property incomes are distinguished: royalties, rents and profits. By royalties, we mean the profits that the owners of exhaustible resources earn in order to keep their capital "crystallised" in the resource mines; rents refer to the income earned by the owners and caused by the differences in fertilities of resource mines; and profits are the income earned by capitalists for using their capital. This distinction is important because the laws that regulate these types of incomes are different.

More formally, let $\boldsymbol{p}_t$, $\boldsymbol{y}_t$, $\boldsymbol{q}_t$ $(t \in N_0)$ denote the prices of commodities, royalties and rents paid to the owners of the resources. Let r_t be the nominal rate of profit at time t, and $\boldsymbol{w}_t$ a bundle of wage goods which is assumed to be exogenously given and constant.

As regards the quantity side, let $\boldsymbol{x}_t$ and $\boldsymbol{s}_t$ $(t \in \mathbb{N})$ be the intensities of processes producing commodities and of processes searching resources, respectively. Let $\boldsymbol{z}_t$ $(t \in \mathbb{N}_0$, the set of all non-negative integers) be the amount of exhaustible resources known at time t. Assume that the annual consumption by non-workers is proportional to $\boldsymbol{\delta}$, a bundle of commodities, which is given and constant. More specifically, the consumption is assumed to be γ units of consumption vector $\boldsymbol{\delta}$, where γ is endogenously determined. Let a vector $\boldsymbol{h}$ be the capacity constraints on extractions, whose elements represent the maximum amount of resources which can be extracted at each time.[2] Finally, the initial amounts of resources and commodities are known and given as $\bar{\boldsymbol{z}}$ and $\boldsymbol{v}$, respectively.

Based on the above assumptions, in the condition of free competition, the following inequalities and equations hold:

$$\mathbf{B}\boldsymbol{p}_{t+1} \leqq (\mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t)(1+r_t) + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}_{t+1} \tag{4.1a}$$

$$\boldsymbol{x}_{t+1}^T\mathbf{B}\boldsymbol{p}_{t+1} = \boldsymbol{x}_{t+1}^T[(\mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t)(1+r_t) + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}_{t+1}] \tag{4.1b}$$

$$\boldsymbol{y}_{t+1} \leqq (1+r_t)\boldsymbol{y}_t \tag{4.1c}$$

$$\boldsymbol{z}_{t+1}^T\boldsymbol{y}_{t+1} = (1+r_t)\boldsymbol{z}_{t+1}^T\boldsymbol{y}_t \tag{4.1d}$$

$$\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} \leqq \mathbf{F}\boldsymbol{p}_t(1+r_t) + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1} \tag{4.1e}$$

$$\boldsymbol{s}_{t+1}^T\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} = \boldsymbol{s}_{t+1}^T[\mathbf{F}\boldsymbol{p}_t(1+r_t) + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1}] \tag{4.1f}$$

[2]It is possible that the capacity constraints on extraction are time-variant due to technological progress, or caused by higher investments made to build more facilities for extraction on resource mines. To simplify the analysis, in this chapter we assume that the capacity constraints on extraction are given and constant. However, this is enough to illustrate how rents can arise on the resource mines.

$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F} \tag{4.1g}$$

$$\boldsymbol{v}^T\boldsymbol{p}_0 = \boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F}\boldsymbol{p}_0 \tag{4.1h}$$

$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T \geqq \boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F} \tag{4.1i}$$

$$[\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T]\boldsymbol{p}_{t+1} = (\boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F})\boldsymbol{p}_{t+1} \tag{4.1j}$$

$$\boldsymbol{z}_0^T \geqq \boldsymbol{x}_1^T\mathbf{C} + \boldsymbol{z}_1^T \tag{4.1k}$$

$$\boldsymbol{z}_0^T\boldsymbol{y}_0 = (\boldsymbol{x}_1^T\mathbf{C} + \boldsymbol{z}_1^T)\boldsymbol{y}_0 \tag{4.1l}$$

$$\boldsymbol{z}_t^T + \boldsymbol{s}_t^T\mathbf{D}_t \geqq \boldsymbol{x}_{t+1}^T\mathbf{C} + \boldsymbol{z}_{t+1}^T \tag{4.1m}$$

$$(\boldsymbol{z}_t^T + \boldsymbol{s}_t^T\mathbf{D}_t)\boldsymbol{y}_t = (\boldsymbol{x}_{t+1}^T\mathbf{C} + \boldsymbol{z}_{t+1}^T)\boldsymbol{y}_t \tag{4.1n}$$

$$\boldsymbol{x}_{t+1}^T\mathbf{C} \leqq \boldsymbol{h}^T \tag{4.1o}$$

$$\boldsymbol{x}_{t+1}^T\mathbf{C}\boldsymbol{q}_t = \boldsymbol{h}^T\boldsymbol{q}_t \tag{4.1p}$$

$$\boldsymbol{z}_0 = \bar{\boldsymbol{z}} \tag{4.1q}$$

$$\sum_{t=0}^{\infty} \frac{\boldsymbol{\delta}^T\boldsymbol{p}_t}{\prod_{\tau=0}^{t-1}(1+r_t)} = 1 \tag{4.1r}$$

$$\gamma > 0, \quad \boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{y}_t \geqq \mathbf{0}, \quad \boldsymbol{z}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_t \geqq \mathbf{0}, \quad \boldsymbol{s}_t \geqq \mathbf{0} \tag{4.1s}$$

Inequality (4.1a) means that no one can obtain extra profits by producing commodities at time t. Equation (4.1b) means that if there is a process incurring extra costs, then this process is not operated at time $t+1$. Inequality (4.1c) means that no extra profits can be obtained by storing resources from t to $t+1$. Equation (4.1d) means that if the storing activity of one resource cannot obtain the nominal rate of profit at time t, then no such resource is available at time $t+1$. Inequality (4.1e) means that no extra profits can be obtained by discovering resources at time t. Equation (4.1f) means that if one searching process incurs extra costs, then this process is not operated at time $t+1$. Inequalities (4.1g) and (4.1i) mean that the amounts of commodities at time t cannot be smaller than the amounts of commodities required by production and consumption at time $t+1$, and equations (4.1h) and (4.1j) mean that if an amount of one commodity at time t is larger than required, then the corresponding price of this commodity is zero. Inequalities (4.1k) and (4.1m) mean that the amounts of resources known at time t plus the resources discovered at time t cannot be less than the amounts of resources utilised to produce commodities and known at time $t+1$, and equations (4.1l) and (4.1n) mean that if the former amount of one kind of resource is strictly larger than the latter, the royalty of this resource is zero. Inequality (4.1o) means that the amounts of extracted resources cannot be larger than $\boldsymbol{h}$.[3] Equation (4.1p) means that if the extraction of one resource i is less than h_i, the ith element of $\boldsymbol{h}$, no rent is paid to the owner of this resource. Equation (4.1q) states that the initial resources are given as $\bar{\boldsymbol{z}}$. Equation (4.1r) is the num$\acute{e}$raire equation. The meaning of inequality (4.1s) is obvious.

Since it is impossible to determine the dynamics of nominal rates of profits in this model, the sequence of nominal rates of profits $\{r_t\}$ is assumed to be given. The given sequence means that the determination of dynamics of r_t is external to the model. However, the given sequence $\{r_t\}$ does not influence the relative actualised prices, because if the sequences $\{\boldsymbol{p}_t\}$, $\{\boldsymbol{y}_t\}$, $\{\boldsymbol{q}_t\}$, $\{\boldsymbol{z}_t\}$, $\{\boldsymbol{x}_{t+1}\}$, $\{\boldsymbol{s}_{t+1}\}$ are a solution to system (4.1) for the given sequence $\{r_t\}$, then the sequences $\{\boldsymbol{p}'_t\}$, $\{\boldsymbol{y}'_t\}$, $\{\boldsymbol{q}'_t\}$, $\{\boldsymbol{z}_t\}$, $\{\boldsymbol{x}_{t+1}\}$, $\{\boldsymbol{s}_{t+1}\}$ are a solution to the same system for the given sequence $\{\rho_t\}$, provided that

$$\boldsymbol{p}'_t = \prod_{\tau=0}^{t-1} \frac{1+\rho_t}{1+r_t} \boldsymbol{p}_t$$

$$\boldsymbol{y}'_t = \prod_{\tau=0}^{t-1} \frac{1+\rho_t}{1+r_t} \boldsymbol{y}_t$$

$$\boldsymbol{q}'_t = \prod_{\tau=0}^{t-1} \frac{1+\rho_t}{1+r_t} \boldsymbol{q}_t$$

Based on the above reasoning, the sequence $\{r_t\}$ is assumed to be constant

[3] The formalisation of capacity constraints on extraction is slightly different from Kurz and Salvadori's (2009, 2011). In Kurz and Salvadori (2009, 2011), the capacity constraints on extraction are represented by the following inequalities:

$$\begin{aligned} \boldsymbol{z}_t^T + \boldsymbol{s}_t^T \mathbf{D}_t &\leqq \boldsymbol{z}_{t+1}^T + \boldsymbol{h}^T \qquad (*) \\ (\boldsymbol{z}_t^T + \boldsymbol{s}_t^T \mathbf{D}_t)\boldsymbol{q}_t &= (\boldsymbol{z}_{t+1}^T + \boldsymbol{h}^T)\boldsymbol{q}_t \end{aligned}$$

These two formalisations are not equivalent; inequality (4.1o) is implied by inequalities (4.1m) and inequality ($*$). Inequality (4.1m) does not consider the situation that there exists a waste of resources, i.e. that the resources are extracted but not used in production. From this point, the formalisation of capacity constraints on extraction in Kurz and Salvadori (2009, 2011) is more accurate. However, the formalisation in this chapter simplifies the analysis.

and $r_t = 0$. Therefore system (4.1) is simplified as follows:

$$(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T)\boldsymbol{p}_{t+1} \leqq \mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t \tag{4.2a}$$
$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T)\boldsymbol{p}_{t+1} = \boldsymbol{x}_{t+1}^T(\mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t) \tag{4.2b}$$
$$\boldsymbol{y}_{t+1} \leqq \boldsymbol{y}_t \tag{4.2c}$$
$$\boldsymbol{z}_{t+1}^T\boldsymbol{y}_{t+1} = \boldsymbol{z}_{t+1}^T\boldsymbol{y}_t \tag{4.2d}$$
$$\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} \leqq \mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1} \tag{4.2e}$$
$$\boldsymbol{s}_{t+1}^T\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} = \boldsymbol{s}_{t+1}^T(\mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1}) \tag{4.2f}$$
$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F} \tag{4.2g}$$
$$\boldsymbol{v}^T\boldsymbol{p}_0 = (\boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F})\boldsymbol{p}_0 \tag{4.2h}$$
$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T \geqq \boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F} \tag{4.2i}$$
$$[\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T]\boldsymbol{p}_{t+1} = (\boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F})\boldsymbol{p}_{t+1} \tag{4.2j}$$
$$\boldsymbol{z}_0^T \geqq \boldsymbol{x}_1^T\mathbf{C} + \boldsymbol{z}_1^T \tag{4.2k}$$
$$\boldsymbol{z}_0^T\boldsymbol{y}_0 = (\boldsymbol{x}_1^T\mathbf{C} + \boldsymbol{z}_1^T)\boldsymbol{y}_0 \tag{4.2l}$$
$$\boldsymbol{z}_t^T + \boldsymbol{s}_t^T\mathbf{D}_t \geqq \boldsymbol{x}_{t+1}^T\mathbf{C} + \boldsymbol{z}_{t+1}^T \tag{4.2m}$$
$$(\boldsymbol{z}_t^T + \boldsymbol{s}_t^T\mathbf{D}_t)\boldsymbol{y}_t = (\boldsymbol{x}_{t+1}^T\mathbf{C} + \boldsymbol{z}_{t+1}^T)\boldsymbol{y}_t \tag{4.2n}$$
$$\boldsymbol{x}_{t+1}^T\mathbf{C} \leqq \boldsymbol{h}^T \tag{4.2o}$$
$$\boldsymbol{x}_{t+1}^T\mathbf{C}\boldsymbol{q}_t = \boldsymbol{h}^T\boldsymbol{q}_t \tag{4.2p}$$
$$\boldsymbol{z}_0 = \bar{\boldsymbol{z}} \tag{4.2q}$$
$$\sum_{t=0}^{\infty}\boldsymbol{\delta}^T\boldsymbol{p}_t = 1 \tag{4.2r}$$
$$\gamma > 0, \quad \boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{y}_t \geqq \mathbf{0}, \quad \boldsymbol{q}_t \geqq \mathbf{0}, \quad \boldsymbol{z}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_t \geqq \mathbf{0}, \quad \boldsymbol{s}_t \geqq \mathbf{0} \tag{4.2s}$$

In order to avoid "the end of the world" scenario, it is assumed that the annually required consumption $\boldsymbol{\delta}$ can be produced by the "backstop technology" which only uses non-exhaustible resources. Formally, the backstop technology, denoted by $(\bar{\mathbf{A}}, \mathbf{0}, \bar{\boldsymbol{l}}_1, \bar{\mathbf{B}})$, is formed by the processes obtained from $(\mathbf{A}, \mathbf{C}, \boldsymbol{l}_1, \mathbf{B})$ by deleting all the processes directly using exhaustible resources, i.e. a process $(\boldsymbol{e}_i^T\mathbf{A}, \boldsymbol{e}_i^T\mathbf{C}, \boldsymbol{e}_i^T\boldsymbol{l}_1, \boldsymbol{e}_i^T\mathbf{B})$ is in $(\bar{\mathbf{A}}, \mathbf{0}, \bar{\boldsymbol{l}}_1, \bar{\mathbf{B}})$ if, and only if, $\boldsymbol{e}_i^T\mathbf{C} = \mathbf{0}$. The remaining processes are denoted by $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \tilde{\boldsymbol{l}}_1, \tilde{\mathbf{B}})$. The existence of backstop technology is summarised as follows:

Assumption 4.2. *There exists a scalar r^* and there are vectors $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$*

which solve the system:

$$\boldsymbol{x}^T(\bar{\mathbf{B}} - \bar{\mathbf{A}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T) \geqq \boldsymbol{\delta}^T \tag{4.3a}$$
$$\boldsymbol{x}^T(\bar{\mathbf{B}} - \bar{\mathbf{A}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T)\boldsymbol{p} = \boldsymbol{\delta}^T\boldsymbol{p} \tag{4.3b}$$
$$\bar{\mathbf{B}}\boldsymbol{p} \leqq [(1+r)\bar{\mathbf{A}} + \bar{\boldsymbol{l}}_1\boldsymbol{w}^T]\boldsymbol{p} \tag{4.3c}$$
$$\boldsymbol{x}^T\bar{\mathbf{B}}\boldsymbol{p} = x^T[(1+r)\bar{\mathbf{A}} + \bar{\boldsymbol{l}}_1\boldsymbol{w}^T]\boldsymbol{p} \tag{4.3d}$$
$$\boldsymbol{x} \geqq \mathbf{0}, \quad \boldsymbol{p} \geqq \mathbf{0}, \quad \boldsymbol{\delta}^T\boldsymbol{p} = 1 \tag{4.3e}$$

Such processes operated at intensity $\bar{\boldsymbol{x}}$, which is obtained from $\boldsymbol{x}^*$ by augmenting it with zeros, will be called "cost-minimising backstop processes" and will be denoted by $(\hat{\mathbf{A}}, \mathbf{0}, \hat{\boldsymbol{l}}_1, \hat{\mathbf{B}})$. The backstop technology and cost-minimising backstop processes are assumed to have the following characteristics.

Assumption 4.3. *The processes in the backstop technology converge to the processes* $(\hat{\mathbf{A}}, \mathbf{0}, \hat{\boldsymbol{l}}_1, \hat{\mathbf{B}})$. *In other words, for each solution of the following system, there exists a natural number* θ^* *such that for each* $t \geqq \theta^*$, *only the processes* $(\hat{\mathbf{A}}, \mathbf{0}, \hat{\boldsymbol{l}}_1, \hat{\mathbf{B}})$ *are operated:*

$$(\bar{\mathbf{B}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T)\boldsymbol{p}_{t+1} \leqq \bar{\mathbf{A}}\boldsymbol{p}_t \tag{4.4a}$$
$$\boldsymbol{x}_{t+1}^T(\bar{\mathbf{B}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T)\boldsymbol{p}_{t+1} = \boldsymbol{x}_{t+1}^T\bar{\mathbf{A}}\boldsymbol{p}_t \tag{4.4b}$$
$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T\bar{\mathbf{A}} + \gamma\boldsymbol{\delta}^T \tag{4.4c}$$
$$\boldsymbol{v}^T\boldsymbol{p}_0 = (\boldsymbol{x}_1^T\bar{\mathbf{A}} + \gamma\boldsymbol{\delta}^T)\boldsymbol{p}_0 \tag{4.4d}$$
$$\boldsymbol{x}_{t+1}^T(\bar{\mathbf{B}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T) \geqq \boldsymbol{x}_{t+2}^T\bar{\mathbf{A}} + \gamma\boldsymbol{\delta}^T \tag{4.4e}$$
$$\boldsymbol{x}_{t+1}^T(\bar{\mathbf{B}} - \bar{\boldsymbol{l}}_1\boldsymbol{w}^T)\boldsymbol{p}_{t+1} = (\boldsymbol{x}_{t+2}^T\bar{\mathbf{A}} + \gamma\boldsymbol{\delta}^T)\boldsymbol{p}_{t+1} \tag{4.4f}$$
$$\sum_{t=0}^{\infty}\boldsymbol{\delta}^T\boldsymbol{p}_t = 1 \tag{4.4g}$$
$$\gamma > 0, \quad \boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_{t+1} \geqq \mathbf{0} \tag{4.4h}$$

Assumption 4.4. *The number of cost-minimising backstop processes is exactly* n*; the matrix* $[\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1\boldsymbol{w}^T]$ *is invertible; the matrix* $[\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1\boldsymbol{w}^T]^{-1}\hat{\mathbf{A}}$ *is* *non-negative, and the eigenvalue of the maximum modulus of the matrix* $[\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1 \boldsymbol{w}^T]^{-1}\hat{\mathbf{A}}$ *is smaller than unity.*

Assumption 4.3 is made in order to separate the problem of "convergence" from the present analysis. The theory that the market prices continue to gravitate to the natural prices determined by the cost-minimising technique is well elaborated by the classical economists, and it is also advocated by some early neo-classical economists like Marshall, Walras and Wicksell. Even though there are debates on how to formally explain the convergence problem in modern classical theory (for instance see a survey by Bellino 2011), it is still legitimate to take it as a reasonable assumption, given the fact that plenty of empirical facts have supported convergence (Petri 2011). Assumption 4.4 is made to simplify the analysis, and it certainly holds if joint production is ruled out.

After setting the basic framework, we will proceed to analyse the dynamics of quantities and prices.

4.3 Economic Dynamics of Quantities and Prices

In this section we investigate the dynamics of both the quantity side and price side of the above model. We are interested in answering the following questions: can there exist paths of quantities (including commodities produced and resources remaining in the ground) and prices (including prices of commodities, royalties and rents of resources) such that the system sustains itself?[4] If there exist such paths, what are the sufficient and necessary conditions?

[4] It is also interesting to investigate whether there can exist a balanced growth path and a set of constant prices for the above model. This problem and the one dealt with in this chapter can be considered as two scenarios based on two "polar assumptions" as put forward by Solow (1959) with respect to his model and Morishima's (1959). It is hard to argue which is more important or interesting. We will discuss the circumstances under which there can exist a set of constant prices in the next section. As regards economic growth, albeit not being dealt with explicitly in this chapter, such growth is nothing but changes in quantities. Hence the problem of economic growth is taken into consideration to some extent.

In order to answer these questions, let us first simplify the analysis and assume that the total amounts of resources are given, and the searching costs are negatively related to the amounts of unknown resources. Since the remaining resources will become ever-diminishing as the searching activities proceed, the searching costs are assumed to be increasing, i.e. $\alpha \in (0, 1)$. Assume that system (4.2) has a solution, and call the processes operated at time t in this solution as the position at time t. The number of possible positions is finite due to the fact the number of processes is finite. Therefore, at least one position is replicated for an infinite time. Because the amounts of resources are given, and the searching costs are increasing, which means that the searching processes cannot be operated for ever, and the vector of the amounts of resources utilised in each position is bounded from below, any position which is replicated for an infinite number of times either includes no processes using exhaustible resources at all, or includes processes using exhaustible resources together with the processes which can produce the consumption vector $\gamma\boldsymbol{\delta}$ without using exhaustible resources. Therefore, we can divide the period from time 0 to infinity into two subperiods: a finite period from time 0 to τ', and an infinite subperiod from time $\tau'+1$ to infinity, on condition that, in the second subperiod, only the backstop processes are actually operated. In addition, if Assumption 4.3 and Assumption 4.4 hold, we can divide the period from time $\tau' + 1$ to infinity into two subperiods: a finite subperiod from $\tau' + 1$ to τ'' and an infinite subperiod from $\tau'' + 1$ to infinity, on condition that, in the second period:

$$\boldsymbol{p}_t = \mathbf{A}^{*t-\tau''}\boldsymbol{p}_{\tau''}$$

$$\boldsymbol{y}_t = \boldsymbol{y}_{\tau''}$$

where $\mathbf{A}^* = [\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1\boldsymbol{w}^T]^{-1}\hat{\mathbf{A}}$. In other words, after τ'', the cost-minimising backstop processes are operated.

The above reasoning suggests that we can investigate the following system as a preliminary step to the analysis of system (4.2).

$$(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T)\boldsymbol{p}_{t+1} \leqq \mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t \quad 0 \le t \le \theta - 1 \tag{4.5a}$$

$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T)\boldsymbol{p}_{t+1} = \boldsymbol{x}_{t+1}^T(\mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t) \quad 0 \le t \le \theta - 1 \tag{4.5b}$$

$$\boldsymbol{y}_{t+1} \leqq \boldsymbol{y}_t \quad 0 \le t \le \theta - 1 \tag{4.5c}$$

$$\boldsymbol{z}_{t+1}^T \boldsymbol{y}_{t+1} = \boldsymbol{z}_{t+1}^T \boldsymbol{y}_t \quad 0 \le t \le \theta - 1 \tag{4.5d}$$

$$\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} \leqq \mathbf{F}\boldsymbol{p}_t + l_2 w^T \boldsymbol{p}_{t+1} \quad 0 \le t \le \theta - 1 \tag{4.5e}$$

$$\boldsymbol{s}_{t+1}^T \mathbf{D}_{t+1}\boldsymbol{y}_{t+1} = \boldsymbol{s}_{t+1}^T[\mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2 \boldsymbol{w}^T \boldsymbol{p}_{t+1}] \quad 0 \le t \le \theta - 1 \tag{4.5f}$$

$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_1^T \mathbf{F} \tag{4.5g}$$

$$\boldsymbol{v}^T \boldsymbol{p}_0 = (\boldsymbol{x}_1^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_1^T \mathbf{F})\boldsymbol{p}_0 \tag{4.5h}$$

$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T \boldsymbol{l}_2 \boldsymbol{w}^T \geqq \boldsymbol{x}_{t+2}^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T \mathbf{F} \quad 0 \le t \le \theta - 2 \tag{4.5i}$$

$$[\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T \boldsymbol{l}_2 \boldsymbol{w}^T]\boldsymbol{p}_{t+1} = (\boldsymbol{x}_{t+2}^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T \mathbf{F})\boldsymbol{p}_{t+1} \quad 0 \le t \le \theta - 2 \tag{4.5j}$$

$$\boldsymbol{x}_\theta^T(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T) \geqq \gamma \boldsymbol{\delta}^T + \gamma \boldsymbol{\delta}^T(\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{A}^* \tag{4.5k}$$

$$\boldsymbol{x}_\theta^T(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T)\boldsymbol{p}_\theta = [\gamma \boldsymbol{\delta}^T + \gamma \boldsymbol{\delta}^T(\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{A}^*]\boldsymbol{p}_\theta \tag{4.5l}$$

$$\bar{\boldsymbol{z}} \geqq \boldsymbol{x}_1^T \mathbf{C} + \boldsymbol{z}_1^T \tag{4.5m}$$

$$\bar{\boldsymbol{z}}\boldsymbol{y}_0 = (\boldsymbol{x}_1^T \mathbf{C} + \boldsymbol{z}_1^T)\boldsymbol{y}_0 \tag{4.5n}$$

$$\boldsymbol{z}_t^T + \boldsymbol{s}_t^T \mathbf{D}_t \geqq \boldsymbol{x}_{t+1}^T \mathbf{C} + \boldsymbol{z}_{t+1}^T \quad 1 \le t \le \theta - 1 \tag{4.5o}$$

$$(\boldsymbol{z}_t^T + \boldsymbol{s}_t^T \mathbf{D}_t)\boldsymbol{y}_t = (\boldsymbol{x}_{t+1}^T \mathbf{C} + \boldsymbol{z}_{t+1}^T)\boldsymbol{y}_t \quad 1 \le t \le \theta - 1 \tag{4.5p}$$

$$\boldsymbol{x}_{t+1}^T \mathbf{C} \leqq \boldsymbol{h}^T \quad 0 \leq t \leq \theta \tag{4.5q}$$

$$\boldsymbol{x}_{t+1}^T \mathbf{C} \boldsymbol{q}_t = \boldsymbol{h}^T \boldsymbol{q}_t \quad 0 \leq t \leq \theta \tag{4.5r}$$

$$\sum_{t=0}^{\theta-1} \boldsymbol{\delta}^T \boldsymbol{p}_t + \sum_{t=\theta}^{\infty} \boldsymbol{\delta}^T \mathbf{A}^{*t-\theta} \boldsymbol{p}_\theta = 1 \tag{4.5s}$$

$$\boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{y}_t \geqq \mathbf{0}, \quad \boldsymbol{q}_t \geqq \mathbf{0} \quad 0 \leq t \leq \theta \tag{4.5t}$$

$$\boldsymbol{z}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_t \geqq \mathbf{0}, \quad \boldsymbol{s}_t \geqq \mathbf{0}, \quad 1 \leq t \leq \theta \tag{4.5u}$$

$$\gamma > 0 \tag{4.5v}$$

In system (4.5), θ is a positive natural number. The above system can be considered as consisting of the first θ steps of system (4.2), on condition that $\boldsymbol{x}_{\theta+1} = \gamma \bar{\boldsymbol{x}}$, which means that, at time $\theta + 1$, the cost-minimising backstop processes are operated and the intensities are such that the cost-minimising backstop processes produce γ units of consumption vector $\boldsymbol{\delta}$. Therefore $\boldsymbol{x}_{\theta+1}^T \mathbf{A} = \gamma \boldsymbol{\delta}^T (\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{A}^{*}$[5] and prices for $t > \theta$ are:

$$\boldsymbol{p}_t = \mathbf{A}^{*t-\theta} \boldsymbol{p}_\theta$$

The searching processes cannot be operated forever, and there is no harm in assuming that the searching activities stop before the backstop processes are operated. Therefore we have $\boldsymbol{s}_t = 0$ for all $t \geqq \theta$.

In what follows we first give a sufficient and necessary condition for the existence of solutions to system (4.5). It can be shown that system (4.5) is equivalent to the following dual problems:

---

[5]Let $\hat{\boldsymbol{x}}^*$ be the intensities corresponding to $(\hat{\mathbf{A}}, \mathbf{0}, \hat{\boldsymbol{l}}_1, \hat{\mathbf{B}})$. $\bar{\boldsymbol{x}}^T \mathbf{A} = \hat{\boldsymbol{x}}^{*T} \hat{\mathbf{A}}$, and $\hat{\mathbf{A}} = (\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1 \boldsymbol{w}^T) \mathbf{A}^*$

$$\hat{\boldsymbol{x}}^* (\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1 \boldsymbol{w}^T - \hat{\mathbf{A}}) = \hat{\boldsymbol{x}}^* (\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1 \boldsymbol{w}^T)(\mathbf{I} - \mathbf{A}^*) = \boldsymbol{\delta}^T.$$

$$\hat{\boldsymbol{x}}^* (\hat{\mathbf{B}} - \hat{\boldsymbol{l}}_1 w^T) \mathbf{A}^* = \boldsymbol{\delta}^T (\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{A}^*.$$

$$\text{Min } \boldsymbol{v}^T \boldsymbol{p}_0 + \bar{\boldsymbol{z}}^T \boldsymbol{y}_0 + \boldsymbol{h}^T \sum_{t=0}^{\theta} \boldsymbol{q}_t$$

s.t.

$$(\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T)\boldsymbol{p}_{t+1} \leqq \mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t \quad 0 \le t \le \theta - 1 \tag{4.6a}$$

$$\boldsymbol{y}_{t+1} \leqq \boldsymbol{y}_t \quad 0 \le t \le \theta - 1 \tag{4.6b}$$

$$\mathbf{D}_{t+1}\boldsymbol{y}_{t+1} \leqq \mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2 \boldsymbol{w}^T \boldsymbol{p}_{t+1} \quad 0 \le t \le \theta - 1 \tag{4.6c}$$

$$\sum_{t=0}^{\theta-1} \boldsymbol{\delta}^T \boldsymbol{p}_t + \boldsymbol{\delta}^T (\mathbf{I} - \mathbf{A}^*)^{-1} \boldsymbol{p}_\theta = 1 \tag{4.6d}$$

$$\boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{y}_t \geqq \mathbf{0}, \quad \boldsymbol{q}_t \geqq \mathbf{0}, \quad 0 \le t \le \theta \tag{4.6e}$$

$$\text{Max } \gamma \equiv \gamma_2 - \gamma_1$$

s.t.

$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_1^T \mathbf{F} \tag{4.7a}$$

$$\boldsymbol{x}_{t+1}^T (\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T \boldsymbol{l}_2 \boldsymbol{w}^T \geqq \boldsymbol{x}_{t+2}^T \mathbf{A} + \gamma \boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T \mathbf{F} \quad 0 \le t \le \theta - 2 \tag{4.7b}$$

$$\boldsymbol{x}_\theta^T (\mathbf{B} - \boldsymbol{l}_1 \boldsymbol{w}^T) \geqq \gamma \boldsymbol{\delta}^T + \gamma \boldsymbol{\delta}^T (\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{A}^* \tag{4.7c}$$

$$\bar{\boldsymbol{z}} \geqq \boldsymbol{x}_1^T \mathbf{C} + \boldsymbol{z}_1^T \tag{4.7d}$$

$$\boldsymbol{z}_t^T + \boldsymbol{s}_t^T \mathbf{D}_t \geqq \boldsymbol{x}_{t+1}^T \mathbf{C} + \boldsymbol{z}_{t+1}^T \quad 1 \le t \le \theta - 1 \tag{4.7e}$$

$$\boldsymbol{x}_{t+1}^T \mathbf{C} \leqq \boldsymbol{h}^T \quad 0 \leqq t \leqq \theta \tag{4.7f}$$

$$\boldsymbol{z}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_t \geqq \mathbf{0}, \quad \boldsymbol{s}_t \geqq \mathbf{0}, \quad 1 \le t \le \theta \tag{4.7g}$$

Lemma 4.1. *If there is a backstop technology, system* (4.5) *has a solution for* $\theta = \theta'$*, if, and only if, the following Assumption 4.5 holds.*

Assumption 4.5. *There are three finite sequences* $\{\boldsymbol{x}_t\}$, $\{\boldsymbol{s}_t\}$, $\{\boldsymbol{z}_t\}$, $1 \leqq t \leqq \theta$, *and a real number* γ, *such that system* (4.7) *holds for* $\theta = \theta'$.

Proof.

Assumption 4.5 states that there exists a feasible solution to the dual. It can be checked that there is such a large real number $\sigma > 0$ that the following sequences

$$\boldsymbol{p}_t = \frac{r^*}{(1+r^*)^{t+1}}\boldsymbol{p}^* \quad (t = 0, 1, \cdots, \theta')$$

$$\boldsymbol{y}_t = \boldsymbol{0} \quad (t = 0, 1, \cdots, \theta')$$

$$\boldsymbol{q}_t = \sigma\boldsymbol{e} \quad (t = 0, 1, \cdots, \theta')$$

are a feasible solution to the primal. Therefore, the duality theorem guarantees that both the primal and dual have optimal solutions with a positive value of γ for $\theta = \theta'$.

Q.E.D.

After showing the condition such that the first θ steps of system (4.2) have a solution, then the problem close at hand is as follows: can there exist paths of quantities ($\{\boldsymbol{x}_t\}$, $\{\boldsymbol{s}_t\}$, $\{\boldsymbol{z}_t\}$) and prices ($\{\boldsymbol{p}_t\}$, $\{\boldsymbol{y}_t\}$, $\{\boldsymbol{q}_t\}$) such that system (4.2) sustains itself? This question will be answered by the following lemma and theorem.

If a solution can be found for a $\theta = \theta'$, then we can find a solution for $\theta = \theta''$ where $\theta'' \geq \theta'$, as is shown by the following lemma.

Lemma 4.2. *If system* (4.5) *has a solution for* $\theta = \theta'$, *then it has a solution for* $\theta = \theta''$, *each* $\theta'' \geq \theta'$.

Proof.

If three finite sequences $\{\boldsymbol{x}'_t\}$, $\{\boldsymbol{s}'_t\}$, $\{\boldsymbol{z}'_t\}$ $(t = 1, 2, \cdots, \theta')$ and a real number γ' satisfy system (4.7), then three finite sequences $\{\boldsymbol{x}''_t\}$, $\{\boldsymbol{s}''_t\}$, $\{\boldsymbol{z}''_t\}$ with $\boldsymbol{x}''_t = \boldsymbol{x}'_t$, $\boldsymbol{s}''_t = \boldsymbol{s}'_t$, and $\boldsymbol{z}''_t = \boldsymbol{z}'_t$ for $t = 1, 2, \cdots, \theta'$, and $\boldsymbol{x}''_t = \gamma'\bar{\boldsymbol{x}}$,

$\boldsymbol{z}''_t = \boldsymbol{z}'_\theta$, and $\boldsymbol{s}''_t = \mathbf{0}$ for $t = \theta' + 1, \theta' + 2, \cdots, \theta''$, and the real number γ' satisfy system (4.7) for $\theta = \theta''$. Based on the same reasoning as in Lemma 4.1, there exists a solution for $\theta = \theta''$, each $\theta'' \geq \theta'$.

Q.E.D.

Assume that there is a natural number θ' such that Assumption 4.5 holds. Then for each $\theta \geq \theta'$ the maximum of the dual problem exists and is positive. Call this maximum γ_θ. For each $\theta \geq \theta'$, define six sequences $\{\boldsymbol{x}_{t\theta}\}$, $\{\boldsymbol{z}_{t\theta}\}$, $\{\boldsymbol{s}_{t\theta}\}$, $\{\boldsymbol{p}_{t\theta}\}$, $\{\boldsymbol{y}_{t\theta}\}$ and $\{\boldsymbol{q}_{t\theta}\}$ as follows: for $t \leq \theta$, $\boldsymbol{p}_{t\theta}$, $\boldsymbol{y}_{t\theta}$ and $\boldsymbol{q}_{t\theta}$ equal the corresponding elements of the optimal solution of the primal problem, and $\boldsymbol{x}_{t\theta}$, $\boldsymbol{z}_{t\theta}$ and $\boldsymbol{s}_{t\theta}$ equal the corresponding elements of the optimal solution of the dual problem, and for $t \geq \theta$:

$$\begin{aligned}
\boldsymbol{p}_{t\theta} &= \mathbf{A}^{*t-\theta}\boldsymbol{p}_{\theta\theta} \\
\boldsymbol{y}_{t\theta} &= \boldsymbol{y}_{\theta\theta} \\
\boldsymbol{q}_{t\theta} &= \boldsymbol{q}_{\theta\theta} \\
\boldsymbol{x}_{t\theta} &= \gamma_\theta \bar{\boldsymbol{x}} \\
\boldsymbol{z}_{t\theta} &= \boldsymbol{z}_{\theta\theta} \\
\boldsymbol{s}_{t\theta} &= \mathbf{0}
\end{aligned}$$

Based on the above analyses, a solution to system (4.2) can be obtained.

Theorem 4.1. *The sequences* $\{\boldsymbol{p}^*_t\}$, $\{\boldsymbol{y}^*_t\}$, $\{\boldsymbol{q}^*_t\}$, $\{\boldsymbol{x}^*_t\}$, $\{\boldsymbol{z}^*_t\}$, $\{\boldsymbol{s}^*_t\}$, *and a real number* γ^* *defined as*

$$\boldsymbol{p}^*_t = \lim_{\theta\to\infty} \boldsymbol{p}_{t\theta} \tag{4.9a}$$

$$\boldsymbol{y}^*_t = \lim_{\theta\to\infty} \boldsymbol{y}_{t\theta} \tag{4.9b}$$

$$\boldsymbol{q}^*_t = \lim_{\theta\to\infty} \boldsymbol{q}_{t\theta} \tag{4.9c}$$

$$\boldsymbol{x}^*_t = \lim_{\theta\to\infty} \boldsymbol{x}_{t\theta} \tag{4.9d}$$

$$\boldsymbol{z}^*_t = \lim_{\theta\to\infty} \boldsymbol{z}_{t\theta} \tag{4.9e}$$

$$\boldsymbol{s}^*_t = \lim_{\theta\to\infty} \boldsymbol{s}_{t\theta} \tag{4.9f}$$

$$\gamma^* = \lim_{\theta\to\infty} \gamma_\theta \tag{4.9g}$$

constitute a solution to system (4.2)*.*

Proof.

If the limits of system (4.9) exist, then it is easily checked that the sequences $\{\boldsymbol{p}_t^*\}$, $\{\boldsymbol{y}_t^*\}$, $\{\boldsymbol{y}_t^*\}$, $\{\boldsymbol{x}_t^*\}$, $\{\boldsymbol{z}_t^*\}$, $\{\boldsymbol{s}_t^*\}$, and the real number γ^* satisfy the equations and inequalities from (4.2b) to (4.2d), and from (4.2f) to (4.2q). What is needed is to prove that these sequences also satisfy inequalities (4.2a) and (4.2e). Suppose the opposite and that for some t, these sequences do not satisfy (4.2a) and (4.2e). Then the first t elements of $\{\boldsymbol{p}_t^*\}$, $\{\boldsymbol{y}_t^*\}$ and $\{\boldsymbol{q}_t^*\}$ cannot be the optimal solution of the primal problem, which contradicts the definitions of $\{\boldsymbol{p}_t^*\}$, $\{\boldsymbol{y}_t^*\}$ and $\{\boldsymbol{q}_t^*\}$. Therefore, sequences $\{\boldsymbol{p}_t^*\}$, $\{\boldsymbol{y}_t^*\}$, $\{\boldsymbol{q}_t^*\}$, $\{\boldsymbol{x}_t^*\}$, $\{\boldsymbol{z}_t^*\}$, $\{\boldsymbol{s}_t^*\}$ and the real number γ^* constitute a solution to system (4.2).

Now let us prove the existence of these limits. Since γ_θ is the maximum for each θ, we know that $\gamma_{\theta+1} \geq \gamma_\theta$, or the sequence $\{\gamma_\theta\}$ is non-decreasing. In addition, $\{\gamma_\theta\}$ is bounded from below (inequality (4.5g)). Therefore this sequence is convergent, and the limit of (4.9g) exists.

Since $\mathbf{0} \leqq \boldsymbol{p}_{t+1}^* \leqq \mathbf{A}^*\boldsymbol{p}_t^*$, $\mathbf{0} \leqq \boldsymbol{y}_{t+1}^* \leqq \boldsymbol{y}_t^*$, and according to the duality theorem, for $\theta = 0$ we have:

$$\gamma^* - \boldsymbol{v}^T\boldsymbol{p}_0^* - \bar{\boldsymbol{z}}^T\boldsymbol{y}_0^* = 0$$

Therefore, the sequences $\{\boldsymbol{p}_t^*\}$, $\{\boldsymbol{y}_t^*\}$ are non-increasing, and the initial values are bounded, which guarantees that the limits of (4.9a) and (4.9b) exist. For $t \geq \theta$, $\boldsymbol{x}_{t\theta} = \gamma_\theta\bar{\boldsymbol{x}}$, (4.5$q$) holds as a strict inequality. Therefore the limit of (4.9c) exists and is equal to zero. The limit of (4.9f) exists because of the definition of the sequence. From the inequalities (4.2k) and (4.2q) we know that the sequence $\{\boldsymbol{z}_t^*\}$ is also non-increasing, and the initial values are bounded, hence its limit exists. The limits of (4.9d) exists, because of the definition of the sequence and the existence of the limit of (4.9g).

Q.E.D.

4.4 The Circumstances of the Existence of Constant Prices

The Hotelling rule implies that the royalties of resources have to increase at the rate that equals the rate of profit, and this seems to further imply that the prices of most, if not all, commodities are bound to change. The latter implication may lead to a belief that the classical economics, which generally focuses on the long-period position or long-period prices, is barren in dealing with the problem of exhaustible resources. This belief is incorrect and misleading. The classical long-period method does not strictly stick to stationary prices. Long-period prices change as the data which determine them change,[6] and in most cases these changes are sufficiently gradual and slow compared with the speed at which market prices gravitate to the long-period, or natural prices. When this premise (the slowness of the changes of long-period prices) does not hold, given the unsatisfactory nature of the marginal approach, "the long-period method appears to be the only acceptable one available at present (Kurz and Salvadori 1995, p.341)". Moreover, the above second implication is not obvious and is not necessarily true. It can be shown that under some well-defined circumstances, the prices of commodities are constant when exhaustible resources exist, as will be shown in this section.

These circumstances have been pointed out by Kurz and Salvadori (2009, 2011). First, the backstop technology is cost-minimising from the beginning, and other processes are not operated. Second, the capacity constraints on extraction are so binding that the processes using exhaustible resources cannot produce the required consumptions alone and backstop technology has to be operated simultaneously. Third, for each exhaustible deposit of resource, another with the same characteristics is discovered, and the searching cost in terms of labour and commodities is always the same. These circumstances

[6] For instance, as Sraffa mentioned, when a plot of less fertile land is taken into cultivation due to an increased demand in an agricultural commodity, "$\cdots$ the output may increase continuously, although the methods of production [hence the prices] are changed spasmodically (Sraffa 1960, p. 88)".

can be represented by the above model.

4.4.1 The Backstop Technology is Cost-Minimising From the Beginning

We first show a situation in which the resources are too costly such that they are neither extracted nor used. For instance, we know that there are plenty of resources in the Arctic Circle, but current technology does not allow us to extract them either due to high costs or due to technological issues. In this situation, the processes of extracting and using resources are dominated by the backstop technology. This situation can be shown as follows.

$$\tilde{\mathbf{B}}\boldsymbol{p}^* < \tilde{\mathbf{A}}\boldsymbol{p}^*(1+r^*) + \tilde{\boldsymbol{l}}_1\boldsymbol{w}^T\boldsymbol{p}^* \tag{4.10}$$

where $\boldsymbol{p}^*$ and r^* are defined by Assumption 4.2. Inequality (4.10) means that the processes using exhaustible resources are not profitable at the rate of profit and prices determined by the cost-minimising backstop processes even if the royalties and rents are zero. Hence no resources will be extracted nor used, and the backstop technology is used immediately. An example like this is given by Kurz and Salvadori (2011). Since the backstop technology converges to the backstop cost-minimising processes, the long-period prices, determined by the latter, are constant. It should again be emphasised that "whether some substance in the ground is, or is not, a resource *cannot generally be defined independently of the rate of profits and the technical alternatives that are available in the system*" (Kurz and Salvadori, 2011, p. 45, emphasis in original.).

4.4.2 The Backstop Technology is Operated From the Beginning Due to Capacity Constraints

In reality, the extraction on each resource mine is usually constrained, and in order to satisfy the required consumption, mines of different unit costs in exploitation, or of different fertilities have to be operated simultaneously. This fact is well observed by the classical economists, especially by Ricardo.

It is also possible that all resource mines which are profitable to extract are all operated but capacity constraints on extractions are so binding that the required consumption cannot be produced by only operating processes which use exhaustible resources, and as a consequence, the backstop technology has to be operated simultaneously. In such a situation, even though the amounts of exhaustible resources decrease and the Hotelling rule applies to the royalties of resources, the prices of commodities are still constant because the owners of the resources are not only paid royalties, but also rents, and because the royalties and rents of resources change in the opposite way such that their summations are constant. This situation is represented as follows.

Assume that Assumption 4.5 holds and a solution to system (4.2) exists. For the optimal solution, the following system holds.

$$\boldsymbol{s}_t^{*T}\mathbf{D}_t < \boldsymbol{x}_{t+1}^{*T}\mathbf{C} \tag{4.11a}$$

$$\boldsymbol{x}_{t+1}^{*T}\mathbf{C} = \boldsymbol{h}^T \tag{4.11b}$$

$$\tilde{\boldsymbol{x}}_{t+1}^{*T}(\tilde{\mathbf{B}} - \tilde{\boldsymbol{l}}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^{*T}\boldsymbol{l}_2\boldsymbol{w}^T < \tilde{\boldsymbol{x}}_{t+2}^{*T}\tilde{\mathbf{A}} + \gamma^*\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^{*T}\mathbf{F} \tag{4.11c}$$

$(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}, \tilde{\boldsymbol{l}}_1, \tilde{\mathbf{B}})$ are processes using exhaustible resources and $\tilde{\boldsymbol{x}}^*$ is the corresponding intensity vector. Inequality (4.11a) means that the resources discovered by the searching activities cannot satisfy the resources needed. Hence the amounts of resources in the ground decrease. Equation (4.11b) means that the extractions of the resources reach their capacity constraints. Inequality (4.11c) means that the processes using exhaustible resources are not able to produce the required consumptions, even though the extractions reach their limits. Hence, the backstop technology has to be operated simultaneously.

Since backstop technology is assumed to converge to the cost-minimising backstop processes, we assume that at some time θ_1, $(\hat{\mathbf{A}}, \mathbf{0}, \hat{\boldsymbol{l}}_1, \hat{\mathbf{B}})$ is operated and the corresponding intensities are $\hat{\boldsymbol{x}}^*$. We also assume that searching activities stop before θ_1. Therefore, after time θ_1, the path of prices $\{\boldsymbol{p}_t\}$ becomes a time-invariant sequence $\{\boldsymbol{p}^*\}$, and equations (4.1a) and (4.1b) are replaced by the following ones:

$$\mathbf{B}\boldsymbol{p}^* \leqq (\mathbf{A}\boldsymbol{p}^* + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t)(1 + r^*) + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}^* \quad t \geqq \theta_1 \tag{4.12a}$$

$$\boldsymbol{x}_{t+1}^{*T}\mathbf{B}\boldsymbol{p}^* = \boldsymbol{x}_{t+1}^{*T}[(\mathbf{A}\boldsymbol{p}^* + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t)(1 + r^*) + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}^*] \quad t \geqq \theta_1 \tag{4.12b}$$

Where $\boldsymbol{p}^*$ and r^* are defined by Assumption 4.2. Let $(\mathbf{A}_t, \mathbf{C}_t, \boldsymbol{l}_{1t}, \mathbf{B}_t)$ represent the actually operated processes at time t ($t \geqq \theta_1$), then we have the following equation.

$$\mathbf{B}_t\boldsymbol{p}^* = (\mathbf{A}_t\boldsymbol{p}^* + \mathbf{C}_t\boldsymbol{y}_t^* + \mathbf{C}_t\boldsymbol{q}_t^*)(1 + r^*) + \boldsymbol{l}_{1t}\boldsymbol{w}^T\boldsymbol{p}^* \quad t \geqq \theta_1 \tag{4.13}$$

From equation (4.13) we can see that $(\boldsymbol{y}_t^* + \boldsymbol{q}_t^*)$ is constant, and even though the royalties increase according to the Hotelling rule, the rents move in the opposite way such that their summations are constant.

The paths of royalties and rents are derived from the following procedure: assume that at time θ_2, the resources will become exhausted. Then at time θ_2 the amount of resources will be very small such that:

$$\boldsymbol{h}^T > \boldsymbol{z}_{\theta_2-1}^T - \boldsymbol{z}_{\theta_2}^T \geqq \tilde{\boldsymbol{x}}_{\theta_2}^*\tilde{\mathbf{C}} \tag{4.14}$$

Therefore at time θ_2, rents are zero because mines are not fully exploited: $\boldsymbol{q}_{\theta_2}^* = \mathbf{0}$, and $\boldsymbol{y}_{\theta_2}^*$ is derived as the differential rents between backstop technology and the processes using exhaustible resources. Taking $\boldsymbol{y}_\theta^*$ as given, royalties at time t can be determined by the Hotelling rule through backward induction, or:

$$\boldsymbol{y}_t^* = (1 + r^*)^{t-\theta_2}\boldsymbol{y}_{\theta_2}^* \quad \theta_1 \leqq t \leqq \theta_2 \tag{4.15}$$

Then taking $\boldsymbol{y}_t^*$ as given, rents $\boldsymbol{q}_t^*$ at each time t can be determined from equation (4.13).

4.4.3 Resources are 'Reproduced' by Labour and Commodities

As stated in previous sections, Ricardo's analysis of exhaustible resources is related to a world with capacity constraints on extractions and searching activities whose costs in terms of commodities and labour are constant. In such a world, the resources are not exactly exhaustible (at least in some periods): they can be found or 'reproduced' by labour and other commodities.[7] This situation can be represented as follows.

Since new resources can always be found and the costs of searching in terms of labour and commodities are always the same, it is appropriate to drop the assumption that the total amounts of resources are known and given, and to assume that the searching costs are time-invariant, i.e. $\alpha = 1$. Otherwise searching activities will cease either due to exhaustion of the resources, or due to increasing costs. Therefore the subscript t can be removed from $\mathbf{D}_t$. In addition, there exist sequences $\{\boldsymbol{x}_t\}$ and $\{\boldsymbol{s}_t\}$ such that the following inequality holds:

$$\boldsymbol{s}_t^T \mathbf{D} \geqq \boldsymbol{x}_{t+1}^T \mathbf{C} \tag{4.16}$$

Inequality (4.16) means that the resources discovered are not less than required. Hence the amounts of resources in the ground at time t are not less than the amounts of resources known at time $t+1$, or we have the following inequality:

$$\boldsymbol{z}_t \geqq \boldsymbol{z}_{t+1} \tag{4.17}$$

In other words, the resources are not exactly exhaustible. Therefore, the storage of resources is not profitable, and system (4.2) is simplified as follows:[8]

[7] This was pointed out by Sraffa in his unpublished Paper and Correspondence, see Kurz and Salvadori (2000b), p. 169.

[8] There exist differences in the model presented in this subsection and the models of renewable resources built by Kurz and Salvadori (1995, Ch. 12, Sec. 3) and Erreygers (2015): in the present model, the costs of "producing" resources are always constant, and there exists capacity constraint on extraction in each resource mine.

$$\mathbf{B}\boldsymbol{p}_{t+1} \leqq \mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}_{t+1} \tag{4.18a}$$
$$\boldsymbol{x}_{t+1}^T\mathbf{B}\boldsymbol{p}_{t+1} = \boldsymbol{x}_{t+1}^T(\mathbf{A}\boldsymbol{p}_t + \mathbf{C}\boldsymbol{y}_t + \mathbf{C}\boldsymbol{q}_t + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}_{t+1}) \tag{4.18b}$$
$$\mathbf{D}\boldsymbol{y}_{t+1} \leqq \mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1} \tag{4.18c}$$
$$\boldsymbol{s}_{t+1}^T\mathbf{D}\boldsymbol{y}_{t+1} = \boldsymbol{s}_{t+1}^T(\mathbf{F}\boldsymbol{p}_t + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}_{t+1}) \tag{4.18d}$$
$$\boldsymbol{v}^T \geqq \boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F} \tag{4.18e}$$
$$\boldsymbol{v}^T\boldsymbol{p}_0 = (\boldsymbol{x}_1^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_1^T\mathbf{F})\boldsymbol{p}_0 \tag{4.18f}$$
$$\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T \geqq \boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F} \tag{4.18g}$$
$$[\boldsymbol{x}_{t+1}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}_{t+1}^T\boldsymbol{l}_2\boldsymbol{w}^T]\boldsymbol{p}_{t+1} = (\boldsymbol{x}_{t+2}^T\mathbf{A} + \gamma\boldsymbol{\delta}^T + \boldsymbol{s}_{t+2}^T\mathbf{F})\boldsymbol{p}_{t+1} \tag{4.18h}$$
$$\boldsymbol{z}_0 \geqq \boldsymbol{x}_1^T\mathbf{C} \tag{4.18i}$$
$$\boldsymbol{z}_0\boldsymbol{y}_0 = \boldsymbol{x}_1^T\mathbf{C}\boldsymbol{y}_0 \tag{4.18j}$$
$$\boldsymbol{s}_t^T\mathbf{D} \geqq \boldsymbol{x}_{t+1}^T\mathbf{C} \tag{4.18k}$$
$$\boldsymbol{s}_t^T\mathbf{D}\boldsymbol{y}_t = \boldsymbol{x}_{t+1}^T\mathbf{C}\boldsymbol{y}_t \tag{4.18l}$$
$$\boldsymbol{x}_{t+1}^T\mathbf{C} \leqq \boldsymbol{h}^T \tag{4.18m}$$
$$\boldsymbol{x}_{t+1}^T\mathbf{C}\boldsymbol{q}_t = \boldsymbol{h}^T\boldsymbol{q}_t \tag{4.18n}$$
$$\sum_{t=0}^{\infty}\boldsymbol{\delta}^T\boldsymbol{p}_t = 1 \tag{4.18o}$$
$$\gamma > 0, \quad \boldsymbol{p}_t \geqq \mathbf{0}, \quad \boldsymbol{y}_t \geqq \mathbf{0}, \quad \boldsymbol{q}_t \geqq \mathbf{0}, \quad \boldsymbol{x}_t \geqq \mathbf{0}, \quad \boldsymbol{s}_t \geqq \mathbf{0} \tag{4.18p}$$

Two questions arise: first, on what condition are there stationary prices to system (4.18)? Second, what condition guarantees that the sequences of price solutions to system (4.18) converge to stationary prices? Discussion of the second question, or the convergence question, is not our concern, and we are only interested in the first. Therefore we assume that the sequences of price solutions converge to the stationary prices, if the latter exist. The following assumption and theorem provide the answer to the first question.

Assumption 4.6. *There exist vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{s}}$ and $\hat{\gamma}$ such that the following inequalities hold:*

$$\hat{\boldsymbol{x}}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \hat{\boldsymbol{s}}^T\boldsymbol{l}_2\boldsymbol{w}^T \geqq \hat{\boldsymbol{x}}^T\mathbf{A} + \hat{\boldsymbol{s}}^T\mathbf{F} + \hat{\gamma}\boldsymbol{\delta}^T \tag{4.19a}$$
$$\hat{\boldsymbol{s}}^T\mathbf{D} \geqq \hat{\boldsymbol{x}}^T\mathbf{C} \tag{4.19b}$$

$$\boldsymbol{h}^T \geqq \hat{\boldsymbol{x}}^T \mathbf{C} \tag{4.19c}$$

Assumption 4.6 means that the technique is productive enough such that, at intensities $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{s}}$, the economy is able to reproduce itself under the capacity constraints of extraction.

Theorem 4.2. *If Assumption 4.6 holds, there exists a solution* $(\boldsymbol{p}^*, \boldsymbol{y}^*, \boldsymbol{q}^*, \boldsymbol{x}^*, \boldsymbol{s}^*, \gamma^*)$ *to the following system:*

$$\mathbf{B}\boldsymbol{p} \leqq \mathbf{A}\boldsymbol{p} + \mathbf{C}\boldsymbol{y} + \mathbf{C}\boldsymbol{q} + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p} \tag{4.20a}$$
$$\boldsymbol{x}^T\mathbf{B}\boldsymbol{p} = \boldsymbol{x}^T(\mathbf{A}\boldsymbol{p} + \mathbf{C}\boldsymbol{y} + \mathbf{C}\boldsymbol{q} + \boldsymbol{l}_1\boldsymbol{w}^T\boldsymbol{p}) \tag{4.20b}$$
$$\mathbf{D}\boldsymbol{y} \leqq \mathbf{F}\boldsymbol{p} + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p} \tag{4.20c}$$
$$\boldsymbol{s}^T\mathbf{D}\boldsymbol{y} = \boldsymbol{s}^T(\mathbf{F}\boldsymbol{p} + \boldsymbol{l}_2\boldsymbol{w}^T\boldsymbol{p}) \tag{4.20d}$$
$$\boldsymbol{x}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}^T\boldsymbol{l}_2\boldsymbol{w}^T \geqq \boldsymbol{x}^T\mathbf{A} + \boldsymbol{s}^T\mathbf{F} + \gamma\boldsymbol{\delta}^T \tag{4.20e}$$
$$[\boldsymbol{x}^T(\mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T) - \boldsymbol{s}^T\boldsymbol{l}_2\boldsymbol{w}^T]\boldsymbol{p} = (\boldsymbol{x}^T\mathbf{A} + \boldsymbol{s}^T\mathbf{F} + \gamma\boldsymbol{\delta}^T)\boldsymbol{p} \tag{4.20f}$$
$$\boldsymbol{s}^T\mathbf{D} \geqq \boldsymbol{x}^T\mathbf{C} \tag{4.20g}$$
$$\boldsymbol{s}^T\mathbf{D}\boldsymbol{y} = \boldsymbol{x}^T\mathbf{C}\boldsymbol{y} \tag{4.20h}$$
$$\boldsymbol{x}^T\mathbf{C} \leqq \boldsymbol{h}^T \tag{4.20i}$$
$$\boldsymbol{x}^T\mathbf{C}\boldsymbol{q} = \boldsymbol{h}^T\boldsymbol{q} \tag{4.20j}$$
$$\boldsymbol{\delta}^T\boldsymbol{p} = 1 \tag{4.20k}$$
$$\gamma > 0, \quad \boldsymbol{p} \geqq \mathbf{0}, \quad \boldsymbol{y} \geqq \mathbf{0}, \quad \boldsymbol{q} \geqq \mathbf{0}, \quad \boldsymbol{x} \geqq \mathbf{0}, \quad \boldsymbol{s} \geqq \mathbf{0} \tag{4.20l}$$

Proof.

Define the following matrices:

$$\mathbb{B} \equiv \begin{bmatrix} \mathbf{B} - \boldsymbol{l}_1\boldsymbol{w}^T & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{l}_2\boldsymbol{w}^T & \mathbf{D} & \mathbf{0} \\ \boldsymbol{\delta}^T & \mathbf{0}^T & \mathbf{0}^T \\ -\boldsymbol{\delta}^T & \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}, \mathbb{A} \equiv \begin{bmatrix} \mathbf{A} & \mathbf{C} & \mathbf{C} \\ \mathbf{F} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}, \boldsymbol{\eta} \equiv \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{y} \\ \boldsymbol{q} \end{bmatrix},$$

$$\boldsymbol{\kappa} \equiv \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \\ \gamma_1 \\ \gamma_2 \end{bmatrix}, \boldsymbol{\xi} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\boldsymbol{h} \end{bmatrix} \boldsymbol{\lambda} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tag{4.21}$$

System (4.20) can be represented as follows:

$$
\begin{aligned}
(\mathbb{B} - \mathbb{A})\boldsymbol{\eta} &\leqq \boldsymbol{\lambda} && \text{(4.22a)}\\
\boldsymbol{\kappa}^T(\mathbb{B} - \mathbb{A})\boldsymbol{\eta} &= \boldsymbol{\kappa}^T\boldsymbol{\lambda} && \text{(4.22b)}\\
\boldsymbol{\kappa}^T(\mathbb{B} - \mathbb{A}) &\geqq \boldsymbol{\xi}^T && \text{(4.22c)}\\
\boldsymbol{\kappa}^T(\mathbb{B} - \mathbb{A})\boldsymbol{\eta} &= \boldsymbol{\xi}^T\boldsymbol{\eta} && \text{(4.22d)}
\end{aligned}
$$

where $\gamma \equiv \gamma_2 - \gamma_1$. System (4.22) can be transformed into two linear programming problems which are dual to each other. The following sequences and Assumption 4.6 guarantee that there exist feasible solutions to the primal and the dual problem:

$$
\begin{aligned}
\boldsymbol{p} &\geq \mathbf{0}\\
\boldsymbol{y} &= \mathbf{0}\\
\boldsymbol{q} &= \sigma\boldsymbol{e}
\end{aligned}
$$

where σ is a very large real number. These conditions ensure that system (4.20) has a solution.

Q.E.D.

We will finish this section with the following remarks.

Remark 4.1. *Ricardo's analysis of exhaustible resources is not inferior to Hotelling's, but is related to a different world with different characterisations, and in a well-defined circumstance, his analysis is correct. Both of their ideas are useful in improving our understanding of the issue of exhaustible resources.*

Remark 4.2. *If capacity constraints on extraction are not binding, i.e.* $\boldsymbol{h}$ *is large enough such that inequality* (4.20*i*) *always holds as a strict inequality, then the model becomes similar to the fixed capital model, and the resources can be treated as machines (suggested by Kurz and Salvadori 1995, Ch. 12). The idea is that the searching processes produce one-year-old machines, and* *the other processes use these one-year-old machines to produce other commodities.*

Remark 4.3. *The difference between the rent theory of land and the above model is that the searching activities require commodities and labour inputs, or resources are 'produced' by labour and commodities. If these inputs are negligible, inequality* (4.20c) *and equation* (4.20d) *do not exist. Then the model falls into the land model in Kurz and Salvadori (1995, Ch. 10).*

4.5 Conclustions

This chapter built an exhaustible resources model, which seeks to make a possible reconciliation between Ricardo and Hotelling, using a dynamic input-output model with classical features. In our model, both resource-searching activities and capacity constraints on extractions exist, and three types of property incomes, profits, royalties and rents, were distinguished explicitly. Both dynamics of quantities and prices were discussed. Based on several assumptions given in the chapter, a sufficient and necessary condition is given for the existence of paths of endogenous variables, which are prices of commodities, royalties, rents, intensities of commodity production and resource-searching processes, and the amounts of exhaustible resources at each time. Following Kurz and Salvadori (2009, 2011), this chapter also discussed the circumstances of the existence of constant prices based on the model presented. It should be re-emphasised that the classical theory is not barren in dealing with exhaustible resources, and that the analysis by the classical economists, especially by Ricardo is not inferior to, but is complementary to Hotelling's. Both ideas can be incorporated into a single framework, and both are helpful in improving our understanding.

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