Behavioral Uncertainty, Learning and Expectations in Macroeconomics

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0.1.1 Definition and epistemic assumptions

Since Keynes' animal spirits, modelling expectations has been a central problem for economists. Adaptive Expectations (Cagan (1956), Nerlove (1958)) was the first paradigm explored. In the simplest version, a representative agent is assumed to form its own (possibly non rational) forecast as a convex combination of their last expectation and the last observation. Formally, adaptive expectation of a certain process y_t evolves according to

$$E_t y_{t+1} = E_{t-1} y_t + \gamma \left(y_t - E_{t-1} y_t \right) \tag{1}$$

where $\gamma \in (0, 1)$ is an exogenous fixed gain. More elaborate versions can include more lagged expectational errors. The problem with this framework is that no optimal individual behavior is implied since (1) is not an efficient correction rule. Hence, agents fail to be efficient.

Nevertheless, if we believe agents are rational, they have to learn something about the working of the economy, they cannot systematically fail efficiency. Using all relevant information, they should be able to correct their forecasts, and so their behavior. The latter argument known as Lucas' critique is at the basis of the mainstream paradigm called rational expectations hypothesis, henceforth REH (Muth 1961, Lucas 1973). REH assumes agents always hold consistent expectations, that is, formally,

$$E_{t-1} y_t = \mathbf{E} \left[y_t | \Omega_{t-1} \right] \tag{2}$$

where $\mathbf{E} \left[y_t | \Omega_{t-1} \right]$ is the mathematical expectations of the stochastic process y_t given the history of the process up to time $t - 1$. (2) implies optimal behavior in stationary environments because what actually matters in stochastic optimization is the first moment of the relevant distribution. But, what are epistemic assumptions underling REH?

Every Macroeconomic stochastic model can be typically reduced or approximated by a linear system in which endogenous variables depend on behavioral (agents' expectations) and exogenous variables. To have consistent forecasts, that is rational expectations, agents have to know the structural relations of economy and the distributions of both exogenous *and* behavioral variables involved. For future reference let's introduce a more precise language. In the context of self-referential models the emergence of a REE occurs with the simultaneous satisfaction of:

a) procedural rationality each agent maintains an optimizing behavior given her beliefs don't contradict *available* information;

b) exogenous consistency each agent knows (or she has correct beliefs about) the model and the exogenous processes involved in the economy;

c) behavioral consistency each agent knows (or she has correct beliefs about) others' expectations.

Procedural rationality is a strength of Bayesian rationality, the kind of rationality implied by a temporary equilibrium, a general economic equilibrium in which agents' beliefs are simply given. The expression *"or she has correct beliefs about"* in the proposition is to stress that what actually matters for the emergence of the equilibrium is the realization of a certain strategy profile[1]. Strictly speaking the concept of knowledge is not necessary.

Differently said, each agent's belief on the outcome of a certain market function (the economic model) has to be consistent given the others' ones (Evans 1983). In this perspective, a rational expectation equilibrium, henceforth REE is a Nash equilibrium of a game in which actions are expectations and the payoffs are inversely proportional to the forecast error variance.

0.1.2 Multiplicity and epistemic foundations problem

As any Nash equilibrium concept, REE presents the well known issue: how do agents coordinate among possible different REE? Or, restating the same question from a normative point of view, which REE should the economist select to calibrate a policy intervention? In Macroeconomics we normally distinguish different kind of REE on the basis of the type and number of variables involved. The fundamental equilibrium is typically expressed (often as a linear approximation) by the smallest number of relevant exogenous variables (McCullum 1983). Bubble solutions are all equilibria in which at least an additional or lagged exogenous variable enters the solution. Finally, sunspot equilibria are

[1] For details about minimal epistemic assumptions for a Nash equilibrium the interested reader is referred to Aumann and Brandenburger (1995).

solutions exhibiting at least one extra variable (normally following a martingale process) not correlated with any relevant exogenous variables. The fundamental solution is normally unique and stable. According to a controversial argument, bubble and sunspot solutions would originate only in a partial equilibrium analysis and would not exist in a general equilibrium perspective[2], because, given market completeness, agents would be able to hedge from collective wrong beliefs. Nevertheless, this argument cannot work in general since, for instance, the probabilistic support of possible sunspots solutions is infinite. Multiplicity is still a big issue for Macroeconomics and it arises within REH paradigm as consequences of its acceptance.

On the other hand, from a more Microeconomic point of view, one can ask under what conditions, if any, the great amount of knowledge assumed by REH paradigm would be maintained by agents. As said REH refers to a potential long run learning of the relevant economic theory held by agents that are supposed able to correct their beliefs. But, according to common sense, rationality should be intended as a procedure, rather than an outcome. *It is not enough to identify such equilibria for which agents' expectations are consistent, but it is necessary to clarify under what rational procedure, if any, agents would be able to correct their beliefs given an initial disequilibrium condition.* The relevant question is when would procedural rationality be sufficient to achieve exogenous consistency and (or) behavioral consistency, that is, when rationality is enough to correct beliefs? What conditions justify epistemic foundations of REE? At this stage learning approaches enters the picture.

0.2 Learning approaches

0.2.1 Learning as solution: the general framework

Modelling explicitly a certain kind of procedurally rational learning has been the main issue of a large body of literature in Macroeconomic dynamics. The reasons of the interest of macroeconomists in learning comes not only from the wide purpose to fully microfound macroeconomics, but rather from the fact that learning is an optimal solution to multiplicity and epistemic assumptions issues.

The general idea is the following. Agents start from a prior disequilibrium, that is, they hold possibly wrong beliefs about the distributions of exogenous and/or behavioral variables. In the language we introduced above, they suffer a lack of exogenous and/or behavioral consistency. Since they are endowed with procedural rationality, they *rationally* exploit all information they have to fill their knowledge gap. In self-referential models

[2] For a deeper reading see Guesnerie 2001

where expectations influence the output, agents can recover new informative data about the actual working of the economy just observing (or deducing) the output generated holding a certain expectation. Learning produces a dynamics of agents' beliefs (in real or notional time) driven by the procedural correct use of forecast errors and possibly converging to a rest point, namely a REE, in which expectations are in equilibrium. This methodology allows to model explicitly an endogenous acquisition of knowledge required to sustain a certain REE and at the same time to produce a natural path-dependent selection among possibly multiple equilibria. In this sense, learning approaches to expectations solve both epistemic foundation and multiplicity problems posed by REH. Of course, learning dynamics can fail to achieve consistency, so that, according to learning approaches a certain REE (even if unique) may be not a relevant equilibrium. In next sections we will briefly review the two main learning approaches to REE, namely eductive and adaptive.

0.2.2 Eductive learning

Eductive Learning (Guesnerie 1992, 2005) takes a game-theoretic point of view. Agents are assumed to maintain procedural rationality and fully exogenous consistency, whereas they lack behavioral consistency. In particular agents hold common knowledge of rationality and of utility functions. Notice that even if such epistemic assumptions are very strong they are not already sufficient to select a unique Nash equilibrium. The question is, under what conditions this would be enough to induce endogenous consistency and hence to justify REE?

Pearce (1984) and Bernheim (1984) elaborated a solution concept named rationalizable set that is the subset of the strategic space resulting from the iterated deletion of strongly dominated strategies. This procedure is based only on common knowledge of rationality and of the game. Whenever the rationalizable set is a singleton it corresponds to the unique Nash equilibrium of the game.

Eductive learning uses this concept in the context of self referential macroeconomic models. Depending on the extent of the impact of aggregate expectation on the economy, the iterated mechanism can converge or not to a REE. In case of multiplicity, local convergence can occur depending on initial conditions. In the latter case great emphasis is placed on common priors (initial conditions), and on the role of policy makers that have to produce and sustain the required common knowledge triggering what is judged to be the best REE.

To enlighten formally the mechanism at the basis of the eductive convergence in a generalized stochastic cobweb model we follow a concise exposition by Evans (2001). Clearing market condition holding ($D(p) = S(p)$), the cobweb model can be restated in the general form

$$p = \alpha + \beta \int_i p_i^e d(i) + \eta \tag{3}$$

where p_i^e is agent i's expectation of actual price level a period before, $i \in (0,1)$, η is a white noise $(0,\delta)$, and α, β are real parameters. The unique REE is $\widetilde{p} = \alpha/(1-\beta) + \eta$. If common knowledge of rationality and of the model hold, than each agent can carry out the following deduction. Suppose each agent have some possibly different initial beliefs about others conjectures on the actual price, and suppose it is common knowledge that every possible belief, and hence the aggregate expectations $p^{e,0}$ as well, is bounded in a neighborhood $\Im(E[\widetilde{p}])$ of $\widetilde{p}$. Formally for each starting conjecture about the aggregate expectation $p^{e,0} \in \Im(E[\widetilde{p}])$, such that $p^{e,0} = E[\widetilde{p}] + \xi = \alpha/(1-\beta) + \xi$ with $\xi \in \mathbb{R}$, each agent deduces a new individual expectation

$$p^{e,1} = \alpha + \beta\left(\alpha/(1-\beta) + \xi\right) = \alpha/(1-\beta) + \beta\xi.$$

Since this is commonly known, a new expectation on the aggregate must be taken in to consideration such that $p^{e,1} \in |\beta| \Im(E[\widetilde{p}])$. Each loop in notional time is formally represented by

$$p^{e,\tau+1} = \alpha + \beta p^{e,\tau} \tag{4}$$

where $\tau \in \mathbb{N}$ indexes notional time. The whole process iterated at infinitum is represented by the recursive formula

$$p^{e,\tau+1} = \alpha \sum_{n=0}^{\tau+1} \beta^n + \beta^{\tau+1} p^{e,\tau} \tag{5}$$

The succession $\{p^{e,\tau}\}^{\infty}$ converges to the REE $p^{e,\infty} = E[\widetilde{p}] = \alpha/\left(1-\beta\right)$ for each arbitrary starting point $p^{e,0} \in \Im(E[\widetilde{p}])$ if and only if $|\beta| < 1$. Figure 1a illustrates possible dynamics of educed price levels starting from a possible minimum price level p_0 (possibly $p_0 = 0$) or a maximum price level P_0 (in the linear case $P_0 = A/B$). It converges to $E[\widetilde{p}]$ if and only if the angular coefficient of the line representing (4), that is β, is constrained between 1 and -1. Otherwise the succession of educed price levels diverges. In the example 1b, higher order expectations diverge from the rational expectation with increasing fluctuations around it.

Therefore, under suitable features, common knowledge of rationality and of the model can generate behavioural consistency of highly endowed agents by a mental process of convergence. It is possible to extend this simple argument in a non-linear model with cobweb timing. In such a case the core result is maintained in a first linear approximation, that is in a local dynamics perspective (Proposition 2, Guesnerie 1992). In such settings the condition $|S'(E_{t-1}p_t)/D'(p_t)| < 1$ is necessary and sufficient to have at least one local

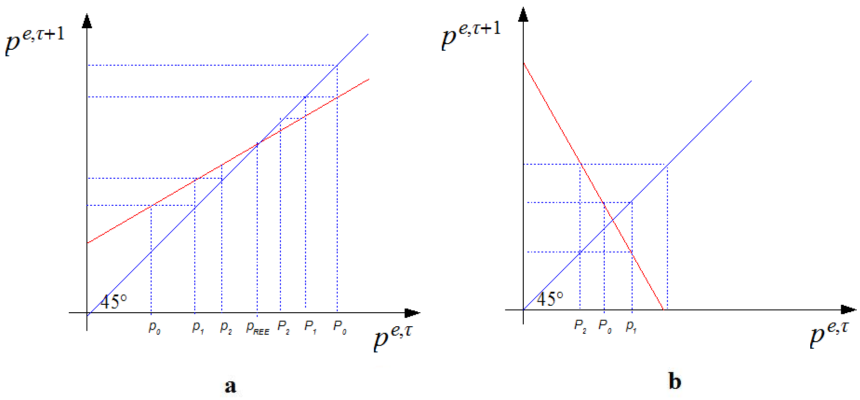

Figure 1: Eductive learning in the cobweb model with different relative slope of demand and supply curve. Case a exhibits convergence, whereas case b does not.

SREE.

0.2.3 Adaptive Learning

The idea in adaptive learning literature is that agents in the model act like econometricians, that is, they have the same knowledge and investigation tools of economists in the real world. In practice, agents are represented by a representative agent that is supposed to update the aggregate expectation according to a structural model of the economy recursively estimated in real time. Agents suffer a lack of exogenous consistency since they don't know the impact of exogenous variables on the economy, but they acknowledge the relevant fundamental variables influencing the equilibrium. The problem of behavioral consistency is instead by-passed by a particular assumption on the way agents form their expectations.

Let us see in practice the key steps of the approach in the context of a simple cobweb-like economy. Consider the following usual reduced form

$$y_t = \phi' x_{t-1} + \beta E_{t-1} y_t + \eta_t \tag{6}$$

where actual output y_t is determined jointly by a vector of exogenous variables x_{t-1}, an aggregate expectation over actual output and a white noise process η_t. In the benchmark case, the representative agent form expectations according to

$$E_{t-1} y_t = b'_{t-1} x_{t-1} \tag{7}$$

where b_{t-1} is a vector of parameters representing subjective beliefs about impulse responses to x_{t-1}. If the structural equation used to forecast accounts for every relevant exogenous variable (as in (7)), agents are said to hold a well specified perceived law of motion. Plugging (7) in (6) we have the actual law of motion

$$y_t = \left(\phi' + \beta b'_{t-1}\right) x_{t-1} + \eta_t \tag{8}$$

which, notice, has a form consistent with (7). This is the case whenever the perceived law of motion is well specified. In particular, the actual dynamics depends only on the so called T-map

$$T_b = \phi' + \beta b'_{t-1}$$

a function going from perceived (estimed) coefficients to actual coefficients. Note that, a key assumption is that, the structural form (7) used by agents encompasses the REE form. Therefore, the recursive effect of the aggregate expectation on actual output is missed, as if actual estimates would not affect future estimates. Nevertheless this is true only in equilibrium, and is not generally the case during transitional dynamics. Referring to such kind of misspecification agents are presented in literature as boundedly rational. No behavioral consistency problem takes place in that agents simply ignore the self referential nature of their forecast problem and the representative agent fiction can be used.

In line with the hypothesis that deep parameters are not time-varying, b_{t-1} is estimated by OLS regression whose recursive formula is[3]

$$b_t = b_{t-1} + t^{-1} R_t^{-1} x_{t-1}(y_{t-1} - b'_{t-1} x_{t-1}) \tag{9a}$$

$$R_t = R_{t-1} + t^{-1}(x_{t-1} x'_{t-1} - R_{t-1}) \tag{9b}$$

known as recursive least square (RLS). Algorithm (9) maintains an adaptive scheme of the form (1). Nevertheless parameters are actually time-varying during convergence process of (9) to its rest point. Any way, this form of misspecification is "possibly temporary", that is, it could vanish asymptotically if expectations converge to RE[4]. Moreover McGough (2000) proves recursion of RLS at hand produces equal convergence condition to the use of a Kalman filter provided agents believe variance on coefficients estimated will vanish not slower than at a certain rate.

The behavior of the dynamic system (8)-(9) is studied by stochastic approximation approach (SSA) whose strict link with the concept of E-stability is extensively shown in Evans and Honkapohja (2001). Here we recall briefly the main ideas. Consider a

[3] Evans and Honkapohja (2001), pag.33, footnote. 4.

[4] Recent literature deals with persistent misspecification when agents neglect some exogenous variable as well. In such cases different equilibria arise than REE, named restricted perception equilibria (see for example Sargent, 1999)

stochastic recursive algorithm (SRA)

$$b_t = b_{t-1} + \gamma_t Q(t, b_{t-1}, X_t) \tag{SRA}$$

where ϕ is a vector of parameters estimates, X_t is the state vector, and γ_t is a deterministic sequence of "gains". The SAA associates an ordinary differential equation (ODE) to the SRA

$$\frac{d\theta_t}{d\tau} = h(\theta(\tau)) \tag{ODE}$$

where $h(\phi(\tau))$ is obtained as

$$h(\phi) = \lim_{t\to\infty} EQ(t, b, X_t) \tag{10}$$

under very general conditions (see Evans and Honkapohja, 2001, Ch. 6-7). Evans and Honkapohja (2001) show $h(b(\tau)) = T_b - b$ where T_b is very simple to obtain in case the perceived law of motion is well specified as in the example at hand. T-map fixed points, for which $T_b(\widehat{b}) = \widehat{b}$, are the best coefficient estimates given available information. Equilibria obtained for $b_{t-1} = \widehat{b}$ correspond to REE. If expectation dynamics driven by (9) algorithm converges to a certain equilibrium, the equilibrium at hand is said to be learnable under RLS. Stability conditions under learning is given on the Jacobian matrix $Jh(b(\tau)) < 0$ and since $h(\phi(\tau)) = T_b - b$ the condition becomes

$$JT_b < 1. \tag{11}$$

Evans and Honkapohja (2001) document extensively variants and analytical results of this class of problems. For the model at hand, the result is easily computed. Agents would achieve exogenous consistency under the adaptive learning dynamics generated by (8)-(9) if and only if $JT_b = \beta < 1$.

In case the perceived law of motion misses same relevant exogenous variable, computing T-map is not immediate. It is still possible to recover T-map from the so called "projected actual law of motion" that is the best (in statistical sense) equilibrium representation with available information. Formally T-map is such that:

$$\mathbf{E}[(x_{t-1})(p_t - T'_\phi x_{t-1})] = 0 \tag{12}$$

where, in case of certainly persistent misspecification, x_{t-1} doesn't contain all variables from which REE depends on. In other words T-map is such that committed error are uncorrelated with available information.

0.2.4 Differences in short

	Eductive (Notional time)	Standard Adaptive (Real time)
Procedural Rationality (Optimizing behavior given available information)	Full	Bounded
Exogenous Consistency (Knowledge of exogenous processes)	Full	Lack (Relevant exogenous variables)
Behavioural Consistency (Knowledge of first order beliefs)	Lack (CK of rationality)	Ignored

Table 1: Learning approaches to REE

Table 1 summarizes methodological differences between two approaches. Eductive Approach assumes full procedural rationality and exogenous consistency, whereas it assume only common knowledge of rationality (that doesn't imply behavioral consistency). Eductive Learning investigates under what conditions this would be enough for agents to hold behavioral consistency. More specifically when the recursive succession of higher order beliefs spanned in notional time converges to REE. Adaptive learning assumes a kind of bounded rationality in that agents employ statistically optimal technique to forecast the course of the economy but they don't recognize the self-referential nature of the economy. For that, the problem of behavioral consistency is simply not considered. Normally agents are assumed to know and perfectly observe the relevant variables affecting the equilibrium, but they ignore the impact of such variables. The general aim with adaptive learning is to spell out conditions to fully achieve exogenous consistency, specifically, whether or not a statistical procedure of beliefs updating in real time, such as (9), converges to correct parameter estimates. In such a case, behavioral consistency is eventually fulfilled in the long run as every agent successfully learns to hold the same rational expectation.

0.3 Aim of the dissertation

Eductive learning assumes agents hold a tremendous amount of information, so that it is hard so maintain it is a positive theory of expectations formation. Adaptive learning is much more appealing from this point of view, as it models agents behaving as real world statisticians. Nevertheless, a general shortcoming of bounded rationality hypothesis is that it often results as an ad-hoc hypothesis. In adaptive learning approach it is not clear why agent should hold that particular form of cognitive limitedness. The issue is more relevant in light of the fact that the findings depend crucially on this hypothesis.

It seems eductive learning does not suffer this problem since it assumes full rationality. *The aim of the dissertation is to understand to which extent adaptive learning approach and convergence conditions are unavoidably linked to the bounded rationality hypothesis.* The issue will be explored in the context of the benchmark case of cobweb-like models.

The work is divided in two parts, each one focusing on a different question. The first part is composed by two papers centred on eductive learning methodology. It investigates whether or not there exists a way to recover adaptive learning convergence conditions with the same eductive learning epistemic assumptions. In the first paper the eductive learning approach is reframed to enlighten the derivation of convergence conditions from application of the iterated deletion of strongly dominated strategies in a proper game-theoretic framework. Eductive learning is generalized to cobweb-like games with a countable number of agents. In particular, it is proven that eductive learning conditions depends not only on relative slope of demand and supply schedules but on the number of active firms in the marked as well. This paper is also preparatory to the second one, that directly tackles the question if eductive learning epistemic assumptions can generate adaptive learning convergence conditions. The answer is yes, provided a slightly different recursion of the iterated deletion of strongly dominated strategies is carried out. Such modification gives rise to the *bi*-iterative rationalizable set whose application in the context of cobweb-like models with a countable number of agents yields adaptive learning convergence conditions irrespective of the number of active firms in the market. The epistemic assumptions sustaining this new iterative solution concept are the same of the classical eductive learning, what changes is the order of the iterative deletion. Specifically, in case of the rationalizable set the whole strategic space is recursively updated at infinity in notional time. Differently, in case of the *bi*-iterative rationalizable set, the iteration is bidimensional in that the strategic space is recursively updated at infinity in notional time *but* for each individual strategic set at time.

The object of the second part of the dissertation is to extend adaptive learning to a truly behavioral uncertainty problem (a typical eductive learning exercise), so overcoming the limits of the bounded rationality hypothesis where interactions are relevant. The third paper investigates the learning dynamics of two agents having non negligible impact on the aggregate expectation, affected by behavioral uncertainty, but perfectly informed about both the exogenous determinants and the self-referential nature of the economy (in this sense they are not bounded rational). Behavioral uncertainty arises in the sense they only have noisy observations of the simultaneous expectation of the other agent. Each agent has to extract the signal of rationality of the other's expectations. Otherwise, because the endogenous and interactive working of the learning algorithm, some equilibria different from REE can arise entailing excess volatility regimes. In sum, I will assume full procedural rationality and exogenous consistency to show how statistical learning can overcome a lack of behavioral consistency in real time.

In the final paper, this scheme governs the arising of endogenous interdependence among institutional forecasters' expectations and it is implemented in a simple monetary Lucas-type[5] model. The most important achievement is to provide a framework in which forecasters learn not only about others' rationality but *also* about the fundamentals. This way it is showed how to merge the theme of learning about others' rationality with the classical theme of learning about fundamentals already developed in standard adaptive learning literature. The paper builds up a simple model that exhibit very standard REE behavior *and,* at the same time, it has the potentiality to trigger persistent and endogenous changes in volatility without relying on any Markov switching or additional aggregate shock.

A common factor to all works is that all the results originate from the breaking of the representative agent fiction. Specifically, we do not introduce particular forms of structural heterogeneity but we maintain homogeneous agents as isolated decision unit and, hence, not informed about the simultaneous expectations of others in the economy. All issues analyzed in this work arise since agents try to solve behavioural uncertainty in the economy.

[5] The reference is to Lucas (1973).

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Part I

Eductive Learning

Chapter 1

Eductive learning revisited

Abstract Eductive learning literature normally investigates rationalizability conditions of rational expectations equilibria in models populated by a continuum of agents. This paper points out that the continuity assumption is not necessary to the argument. Eductive learning approach is reshaped and generalized in the context of Cobweb-like models with a countable set of agents. Convergence requirements involve restrictions also on the number of active firms in the market as well as on the relative slope of demand and supply curves. Usual eductive learning conditions are obtained in the limit of an infinite number of agents. The paper provides a simple model as example to discuss the economic implications of the findings. In general, barriers to entry can make markets with perfect competition cost structure eductively stable whenever they would not naturally be.

1.1 Introduction

Rational expectations equilibria Muth (1961) occur whenever agents are able *in average* to rationally exploit relevant information. This means that what is required to be "rational" in the Muthian concept are not necessary agents. The average expectation can be still "rational" even if agents commit i.i.d. systematic errors. Therefore "rational" is in a certain sense related to the aggregate behavior rather than to the individual one. Nevertheless a game-theoretic literature has developed the concept of REE on the basis of individual rationality from a strict Microfounded interpretation. When agents are *all* required to be fully rational, the REE is a Nash equilibrium of a game in which agents have to coordinate on the same expectation, the rational one, in order to hold a consistent expectation (for a version of this statement see Evans (1983)).

To justify emergence of a particular REE (as any Nash equilibrium) from a disequilibrium initial condition is not trivial at all. One can assume the equilibrium is "focal" in the sense of Schelling (1960); if agents share the common belief that a particular REE is normally achieved, they actually exclude rationally admissible doubts that somebody will fail to conform. One other possibility is that, given a common prior, agents coordinate thanks to a correlated signal as explained in Aumann (1987). Nevertheless, both justifications transcend individual rationality, specifically the strict content of Savage's (1956) axioms. Moreover, common knowledge of individual rationality is generally neither enough nor sufficient to trigger convergence of agents' beliefs on the realization of a particular Nash. Aumann and Brandenburger (1995) clarify minimal epistemic requirements for the emergence of a Nash are common knowledge of first-order beliefs besides *mutual* knowledge of the game and of individual rationality.

Eductive learning approach (Guesnerie (1992, 2005)) investigates conditions under which a strength of mutual knowledge of rationality can substitute common knowledge of agents' first-order beliefs to pin down a Nash equilibrium. This happens whenever common knowledge of rationality and of the model are assumed *and* this is enough to select a singleton in the strategic space. That is, whenever a Nash (or a particular REE, in our case) coincides with the rationalizable set, the subset of the strategic set resulting from iterated deletion of strongly dominated strategies, henceforth IDSDS (Bernheim 1984, Pearce 1984).

Eductive learning approach, as generally presented, takes explicitly the perfect competition approximation assuming a continuum of agents. This is a natural assumption given REE are generally defined in setting where agents are infinitesimally small. According with such a perspective, since the power to displace the aggregate expectation is negligible, agents are basically expectations takers. This means that they don't have any strategic motive to expect something different from the aggregate expectation. Accordingly, IDSDS is implemented with reference to best expectations functions responding to a state of the word, the aggregate expectations on price, rather then on a profile of individual expectations. Approximating a profile of strategies to a state of the word is the same basic insight making perfectly competitive equilibria being asymptotic approximation of equilibria of strategic games in which market power vanishes in the limit.

This paper reframes the eductive approach in a more game-theoretic perspective and extends it to a countable set of agents in the context of Cobweb-like models. The main argument builds up on the point that the assumption of a continuum of agents is not necessary to the eductive analysis. As first step, the paper introduces a simple Cournot model yielding a reduced form that encompasses the Cobweb one. Since the reduced form is obtained from aggregation over a countable set of agents, the feedback of aggregate expectation on actual price is expressed as an explicit function of the number of suppliers in the market. The model is provided as example in order to make crystal clear the connection between the reduced aggregate form usually considered and agents' individual strategic problem.

To this aim, the Cobweb expectations game is defined as the game derived from the original Cournot game, whose strategies are defined as suppliers' beliefs on expected price and whose equilibrium corresponds to all suppliers holding the rational expectation. The notion is useful in that eductive learning actually refers to rationalizability of the equilibrium in the Cobweb expectations game (expressed as a profile of expectations on prices) and not of the one in the Cournot model (expressed as a profile of supplied quantities).

After a brief introduction to the definition of rationalizable set, the paper shows how the perfect competition approximation is used in the derivation of standard eductive learning conditions. The classical eductive learning analysis is reshaped and carried out as a multidimensional dynamics in the strategic space rather than a unidimensional dynamics in the aggregate expectation space.

The framework makes more intuitive and straight the following generalization in Cobweb-like models with a finite countable number of agents. The result is obtained just running the iterated deletion of strongly dominated strategies directly on the Cobweb expectations game where individual best response functions are explicitly expressed as function of expected opponents' strategic profile instead of expected aggregate expectation.

The main finding is that eductive learning conditions involves not only requirements on the relative slope of demand and supply schedules, but also on the number of suppliers in the market. Moreover such conditions result to be the same ones of Guesnerie (1992) in the limit of an infinite number of agents. In other words, classical eductive learning conditions can be recovered computing the limit for an infinite number of agents directly on convergence conditions rather than on best reply functions considered in the IDSDS (as usually done according to the classical approach).

The paper concludes with a brief discussion of the economic implications of the finding on the model presented as example. In particular, given conditions on the slope of both demand and supply curves, it is possible to identify the maximum number of firms active in the market such that the rational expectation equilibrium is rationalizable. Therefore, eductive convergence conditions would entail an entry barrier to the market as one may want to interpret eductive instability of the market as a sufficient disincentive to entry. This would happen even if the industry presents, as in the example at hand, perfectly competitive structure.

1.2 Model

1.2.1 A simple Cournot model

Setting. Consider an economy populated by a countable number of suppliers $I = \{1, ..., s, ...N\} \subset \mathbb{N}$ and an infinite number of consumers. The former make production decisions (quantity) a period before the product is sold and the latter decide if and how much to buy once the total quantity is supplied. The large number of consumers makes them to be price takers. The inverse demand function is described by

$$P_t = k - \alpha D_t + \eta_t \tag{1.1}$$

with P_t labeling price, D_t demanded good, with k and α real parameters and η_t an i.i.d. centred normally distributed stochastic shock. All variables are indexed with time. Market clearing conditions implies

$$D_t = S_t \equiv \sum_{i \in I} q_{i,t-1} \tag{1.2}$$

where S is supplied quantity resulting from aggregation of individual supplied quantities over agents, denoted by q_i as quantity produced by agent i a period before. Each producer has to solve the following problem

$$\max_{q_{i,t-1}} \pi_{i,t}\left(P_t, q_{i,t-1}\right) = P_t q_{i,t-1} - cN q_{i,t-1}^2 - d_{i,t-1} q_{i,t-1} \tag{1.3}$$

where π_i is the profit of suppliers i, c and d_i are parameters shaping respectively the quadratic and linear cost components. In particular the former may reflect crowding effects due to the total number of suppliers in the industry, whereas the latter is a firm specific stochastic component normally i.i.d. distributed with mean d and variance δ_d.

Temporary equilibria. The price P_t is unknown at the date producer i has to set $q_{i,t-1}$. Therefore, the *temporary* optimal individual and aggregate supplied quantities, namely $q_{i,t-1}^*$ and S_{t-1}, are functions of individual subjective (possibly wrong) expectations held at time $t-1$ of current price at time t, labelled[1] $E_{t-1}^i P_t$ with $i \in I$. Solution to suppliers' maximization problem is given by

$$q_{i,t-1}^* = \frac{k - \alpha E_{t-1}^i S_{-i,t} - d_{i,t-1}}{2\left(\alpha + Nc\right)} \tag{1.4}$$

[1]Henceforth expectation of agent i at time $t-1$ on x is written as $E_{t-1}^i x$.

with

$$E_{t-1}^{i}S_{-i,t} \equiv \sum_{j\in I-\{i\}} E_{t-1}^{i}q_{j,t-1}$$

denoting expectations of agent i on residual demand at time t. Given knowledge of (1.1) and (1.2), we can rewrite (1.4) as

$$q_{i,t-1}^{*} = \frac{E_{t-1}^{i}P_t - d_{i,t-1}}{\alpha + 2Nc}. \tag{1.5}$$

that is a function of individual expectation on aggregate price (see appendix for details). At this stage is more evident what is the role of assuming that quadratic costs grow with the same multiplicity of active firms in the market. This deals with aggregation of countable number of agents. The hypothesis is sufficient to maintain infinitesimal individually supplied quantities at any expected price in the limit of $N \to \infty$. Nevertheless, the aggregate supplied quantity varies in accordance with the aggregate expectation. Dropping this assumption (in case of infinite agents) the supply schedule would exhibit infinite elasticity at the point in which price equals the average of marginal costs (see note 4). This means that small deviations of the aggregate expectation from the rational one would imply infinite changes in individual supplied quantity and, this way, indeterminacy of a rationalizable strategic profile (eductive instability) obtains. Similar finding has been stressed by Bernheim (1984) and more generally by Basu (1991) with reference to a Cournot game with linear costs, fixed linear demand curve and infinite suppliers. On the contrary, Borges and Jansenn (1995) showed how a different replication rule of both supply and demand would allow "large" Cournot game to have rationalizable Nash equilibrium, provided the Cobweb property is satisfied for the dynamics generated by IDSDS mechanism in the quantities space. Here, no special replication rule is assumed, nevertheless, in the perfect competition scenario, extreme instability behavior is avoided given individual supplied quantities is kept infinitesimal at any expected price by quadratic costs trend.

According to (1.1),(1.2) and (1.5) there exists a map going from aggregate expected price to actual price given by

$$P_t = k + \frac{\alpha N d_{N,t-1}}{\alpha + 2Nc} - \frac{\alpha N}{\alpha + 2Nc} E_{t-1}P_t + \eta_t \tag{1.6}$$

where $d_{N,t-1} = (1/N)\sum_{i=1}^{N} d_i$ and

$$E_{t-1}P_t \equiv \frac{1}{N}\sum_{i\in I} E_{t-1}^{i}P_t \tag{1.7}$$

is the aggregate (average) expectation. Equation (1.6) determines a temporary equilibrium, that is, a price satisfying market clearing condition (1.2) *given* possibly "wrong" aggregate expectation on the actual market price.

The Rational Expectations Equilibrium. The rational expectations equilibrium is a particular case of temporary equilibrium for which aggregate expectations are consistent forecasts of the actual price.

Definition 1 *Rational expectations equilibria arise whenever aggregate expectations are rational in the sense of Muth (1961), that is*

$$\{E_{t-1}P_t\}_{t=0}^{\infty} = \{\mathbf{E}_{t-1}P_t\}_{t=0}^{\infty} \tag{1.8}$$

where $\mathbf{E}_{t-1}$ is the mathematical expectation of price at time t conditional to objective information available at date $t-1$.

In particular (1.8) and (1.6) imply that

$$\overline{P}_N = k - \alpha \frac{N\left(k - d_{N,t-1}\right)}{\left(N+1\right)\alpha + 2Nc} \tag{1.9}$$

is the unique aggregate rational expectation compatible with non negativity of prices and $P_t^* = \overline{P}_N + \eta_t$ is the unique REE[2]. Of course, in this model, temporary equilibria and the REE depend crucially on the number of firms operating in the market. Notice also that idiosyncratic negative mark-up are not excluded. Nevertheless one can think sunk costs prevent firm to exit from the market in case of expected idiosyncratic loss (see details in appendix). In the limit of the competitive equilibrium $N \to \infty$ the unique rationally expected price is

$$\lim_{N\to\infty} \overline{P}_N = \frac{\alpha d + 2kc}{\alpha + 2c}$$

being $\lim_{N\to\infty}\left(d_{N,t-1}\right) = d$. In this case each firm produces an infinitesimally small quantity, so that, no one is able to influence market outcome.

Change of coordinates. Before to go forward, let us write (1.6) in a more convenient form. From now onward we will refer to

$$p_t = \beta_N E_{t-1} p_t + \eta_t \tag{1.10}$$

[2] From (1.4) we can equally solve for simmetry

$$q_i^* = \frac{k-d}{\left(N+1\right)\alpha + 2Nc}$$

and then substituting in (1.1) and (1.2) one can get equilibrium directly the REE.

where[3] $p_t \equiv P_t - \overline{P}_N$ and $\beta_N \equiv -\alpha N/(\alpha + 2Nc)$. The rational expectation is now expressed as $E_{t-1}p_t = \mathbf{E}_{t-1}p_t = 0$. Reduced form above for $\lim_{N\to\infty}\beta_N \equiv \beta$ is the typical of the popular Cobweb model, that is here generalized as to a countable number of agents. It also encompasses the well known Lucas (1973) demand supply model with the only difference of the sign of β. To make our dynamic analysis consistent with that and other first-order self referential models we will abstract from the sign of β.

The Cournot model above has the aim to provide the simplest possible Microfounded model yielding the popular Cobweb model in its exact approximation for an infinite number of agents. Nevertheless, the main findings of this paper do not depend of the specific form of the model. The analysis can be referred to a generic economy having a Cobweb-like reduced form with a countable number of agents like (1.10). In the final part of this paper we will come back to the economic issues raised by the findings, in the context of the model just presented.

1.2.2 The Cobweb expectations game

In this section we will formally define the coordination game underlying the emergence of the unique Microfounded REE. Let's start observing that the optimal individual solution to the supplier's maximization problem (1.3) in the Cournot model is obtained from (1.5) assuming producer i has rational expectations, that is

$$E^i_{t-1}p_t = \mathbf{E}_{t-1}p_{N,t} \tag{1.11}$$

according to ordinary stochastic maximization algebra rules. As condition (1.11) is not satisfied the agent would suffer a systematic loss of efficiency and welfare. In particular, assuming the solution of the maximization problem is an internal point, the loss will be a convex function whose minimum has to be in correspondence of (1.11). A good representation of suppliers' loss function is

$$\Lambda_{i,t-1} = \mathbf{E}_{t-1}\left(p_t - E^i_{t-1}p_t\right)^2 \tag{1.12}$$

so that the suppliers' loss minimization problem has exactly the unique solution in (1.11). Since actual price moves according to (1.10) agents' loss minimization problem embodies a strategic game.

Suppliers play a one shot game at time $t-1$ in which individual strategies are individual supplied quantities mapped one to one from individual expectations belonging to

[3] In sake of semplicity $p_t(N)$ will be often written as p_t, but don't forget temporary eqiulibria always depends on N!

a given interval centered on the REE, namely $E^i_{t-1}p_t \in \Im(0)^i \subset \mathbb{R}$. The pay-off functions are given directly by (1.12) as functions of a profile of individual expectations. So now we can define the Cobweb expectation game as follows

Definition 2 *The Cobweb expectation game associated with the Cournot model is a one shot game represented by the triple* $\Gamma_N \equiv \{I, \Im(0)^N, \{\Lambda_{i,t-1}\}_I\}$ *defined by a set* I *of agents* i*, a strategic space* $\Im(0)^N \equiv \prod_{i\in I} \Im(0)^i$ *and a set of symmetric payoff functions* $\{\Lambda_{i,t-1}\}_I : \Im(0)^N \rightarrow \mathbb{R}^N$ *going from a profile action to a vector of real numbers where deviations from the fundamental price* p_t *move according to (1.10). By convention* $\lim_{N\to\infty} \Gamma_N \equiv \Gamma$.

The label "Cobweb expectations game" is to remark that its reduced form encompasses the Cobweb one for any countable number of active firms. In defining the expectations game we just exploited the one to one correspondence between individual expectations and individual actions given by (1.5). To compute the equilibrium of the expectation game, condition (1.11) has to be satisfied for each individual where price process is entailed by (1.10) and aggregate expectation is given by (1.7)[4].

The equilibrium of the expectation game is given as an expectations profile satisfying

$$E^i_{t-1}p_t = \beta_N \frac{1}{N} \sum_{j\in I} E^j_{t-1} \quad \forall i \in I \tag{1.13}$$

specifically, this implies $E^i_{t-1}p_t = E_{t-1}p_t = 0$, $\forall i \in I$, that is, every agent maintains rational expectations. Notice that the equilibrium of the Cobweb expectations game is symmetrical, but the equilibrium of the Cournot model is not, due to heterogeneity in $d_{i,t-1}$ present in (1.5).

Remark 3 *The equilibrium of the Cobweb expectations game is a special case of (1.8) where all agents are required to be rational in that they solve their stochastic maximization problem (1.3) according to (1.5) and (1.11).*

Nevertheless, expectations different from the REE prescription $E^i_{t-1}p_t = 0$ can be rationally justified at an individual level by the individual belief of the aggregate expectation can deviate from the rational one. In particular since opponents' simultaneous strategic choices are unknown, agents act given their beliefs on others' expectations. According to the only individual rationality they have to form expectations as function of their beliefs on opponents' strategic choices. Applying individual expectation operator E^i_{t-1} to (1.13) we obtain an expression for the individual best reply function[5] in the

[4] Note that expectation are deterministic data at time $t-1$, hence $\mathbf{E}_{t-1}E^i_{t-1}p_t = E^i_{t-1}p_t$.

[5] From here onward E^{i*}_{t-1} denotes a function whereas E^i_{t-1} just a point in the expectation space.

expectations game given by

$$E_{t-1}^{i*}p_t = \frac{\beta_N}{1-\beta_N/N}\left(\frac{1}{N}\sum_{j\in I-\{i\}} E_{t-1}^{j}p_t\right). \tag{1.14}$$

Eductive learning analysis has the task to assess when is common knowledge of individual rationality enough to exclude beliefs about opponents' expectations that deviate from the unique REE prescription. In such cases common knowledge of rationality is sufficient to justify rational expectations.

1.3 Eductive Learning in the Cobweb expectations game

1.3.1 The rationalizable set

Eductive learning assessment of REE is an application of the game theoretic solution concept of rationalizable set. What follows provides an introduction to it. For a much more detailed treatment the interested reader can refer to Bernheim (1984) and Pearce (1984).

Agents' view of other agents' choices is uncertain. Deviations from a particular equilibrium prescription can be rationally justified as best responses to others' deviations from the same equilibrium prescriptions that in turn can be rationally justified by a similar recursive argument at infinitum. This is the starting observation at the basis of the concept of rationalizable set as exposed in Bernheim's original paper. The rationalizable set is a subset of the strategic space, such that, every individual strategy belonging to it is rationally justified for at least a profile of opponents' strategies belonging to the same set. More specifically, this game theoretic solution concept is constructed through the iterated deletion of strongly dominated strategies (IDSDS). Formally, consider the Cobweb expectations game $\Gamma_N \equiv \{I, \Im(0)^N, \{\Lambda_{i,t-1}\}_I\}$. Index notional time steps, that is, the iteration of the deletion mechanism, by τ. The precise rule of the IDSDS mechanism is given by

$$\begin{aligned}\Im(0)^i_\tau = \Im(0)^i_{\tau-1} - \{&E^i_{t-1,\tau,1} \in \Im(0)^i_{\tau-1} : \forall E^{-i}_{t-1,\tau,1} \in \Im(0)^{-i}_{\tau-1},\\ &\exists\ \widehat{E}^i_{t-1,\tau,1} \in \Im(0)^i_{\tau-1} : \Lambda_{i,t-1}(\widehat{E}^i_{t-1,\tau,1}, s_{-i}) > \Lambda_{i,t-1}(E^i_{t-1,\tau,1}, E^{-i}_{t-1,\tau,1})\}\end{aligned}$$

for each $i \in I$, where $\Im(0)^{-i} \equiv \prod_{j\neq i} \Im(0)^j$. The rationalizable set of the game $R(\Gamma) \subseteq \Im(0)^N$

is obtained as

$$R(\Gamma_N) \equiv \prod_{i \in I} \left(\bigcap_{\tau=0}^{\infty} \Im(0)_{\tau}^{i} \right).$$

The rationalizable set is the finest refinement of the strategic set justified by common knowledge of individual rationality (and of the game). The solution concept of Nash equilibrium corresponds to the rationalizable set if and only if the latter is a singleton. In particular the eductive argument is based on the concept of strongly rational expectations equilibrium (SREE). Below we report the definition of SREE given by Guesnerie (1992).

Definition 4 *A strongly rational-expectations equilibrium (SREE) is a rational expectations equilibrium that is the unique rationalizable equilibrium of the Cobweb expectations game.*

Evans and Guesnerie (1993) use with the same meaning of SREE the expression "eductively stable equilibrium" or "stable under eductive learning". Each rationalizable REE is stable in the sense that beliefs of others will deviate from the REE prescriptions will be rationally excluded at same point of the IDSDS. Hence, given common knowledge of rationality and of the game, a REE is eductively stable if and only if all individual expectations will converge (in notional time) to REE independently on initially believed displacements from it. In other words, the existence of a SREE can justify convergence of players on this particular outcome through a process of mental reasoning as if they were game theoretic economists knowing all the objective features of the problem. Eductive learning literature investigates conditions under which the requirement that agents' beliefs in the Cobweb expectations game belong to the rationalizable set is enough to pin down a unique strategy profile, namely the unique REE of the Cobweb expectations game. More formally said, the aim is to spell out conditions under which $\Re(\Gamma_N) = \{0\}$.

1.3.2 The perfect competition perspective

To enlighten the mechanism at the basis of classical eductive learning in line with Guesnerie (1992)'s approach, it is useful to straightway define the following.

Definition 5 *The perfect competition approximation of individual best replies of the Cobweb expectations game (1.14) is given by*

$$\lim_{N \to \infty} E_{t-1}^{i*} p_t = \beta E_{t-1} p_t. \tag{1.15}$$

In words, given a large number of agents, the individual effect on the aggregate is negligible. Guesnerie (1992)'s setting uses implicitly (1.15). More precisely, in Guesnerie's original paper it is assumed a continuum of agents, so that the approximation seems to be unavoidable. Nevertheless what matters for this result is negligibility of individual contribution to the aggregate expectations. The disappearance of others' expectations (actions) in the righ-hand-side of relation (1.15) makes best replies to be reaction functions to a state of nature instead of to a profile of strategies. Using the perfect competition approximation, the marginal individual influence on aggregate expectations is neglected and so any strategic motives entailed by it. Best expectations functions are so defined as functions of an expected aggregate expectation. This framework entails the following IDSDS.

Let $\Im(0)^N_{\tau-1} \equiv \prod_{i\in I} \Im(0)^i_{\tau-1}$ be the strategic set after the $k-1$-th. step of the IDSDS algorithm. Consider the aggregation mapping $E_{t-1}p_t : \Im(0)^N_{\tau-1} \to \Im(0)_{\tau-1}$ given by (1.7) (and the appropriate change of coordinates). Also consider all individual best replies given by (1.15) as composing a multidimensional mapping $\lim_{N\to\infty} E^{I*}_{t-1}p_t : \Im(0)_{\tau-1} \to \Im(0)^N_\tau$ where $E^{I*}_{t-1}p_t \equiv \left[E^{1*}_{t-1}p_t \cdots E^{i*}_{t-1}p_t \cdots E^{N*}_{t-1}p_t\right]'$ is the $N \times 1$ vector of individual best expectations functions. From composition of these two mappings it is possible to build a new mapping $T \equiv E^{I*}_{t-1}p_t \circ E_{t-1}p_t : \Im(0)^N_{\tau-1} \to \Im(0)^N_\tau$ formally defined by

$$E^I_{t-1,\tau}p_t = \mathbf{B}E^I_{t-1,\tau-1}p_t \tag{1.16}$$

with

$$\mathbf{B} \equiv \beta \, \frac{1}{N} \, \mathbf{1}_N \tag{1.17}$$

where $\mathbf{1}_N$ is an $N \times N$ matrix of ones. This mapping gives the strategic set $T(\Im(0)^N_{\tau-1})$ including only rationally justified strategies for $\Im(0)^N_{\tau-1}$, so the ones that are not deleted by IDSDS at stage τ. Therefore $\Im(0)^N_\tau \equiv T^{\ \tau}(\Im(0)^N)$ is the set of all τ-order rationally justified agents' beliefs. Hence, the rationalizable set is given by $R(\Gamma) \equiv \lim_{\tau\to\infty} T^{\ \tau}(\Im(0)^N)$. Common knowledge of the model and of rationality implies agents to choose an individual strategy belonging to the rationalizable set. In case the rationalizable set is a singleton, agents holding common knowledge of rationality and of the model, will coordinate on the unique REE. Therefore, the Nash of the Cobweb expectations game (that is the REE of the Cobweb model) is a SREE if and only if $\lim_{\tau\to\infty} T^{\ \tau}(\Im(0)) = \{0\}$.

Rationalizability of the expectations Cobweb game are obtained just from dynamic properties of T. REE is eductively justified if and only if T is a contracting map. For T to be a contracting map, eigenvalues of the matrix $\mathbf{B}$ have to be inside the unit circle. To make easy computation of eigenvalues for a general class of matrixes to which $\mathbf{B}$ belongs to, let's state the following lemma.

Lemma 6 *Consider a generic $n\times n$ matrix $\mathbf{A}_n \equiv a(\mathbf{1}_n + b\,\mathbf{I}_n)$, where $\mathbf{1}_n$ is an $n\times n$ matrix of ones, $\mathbf{I}_m$ is the unit matrix of dimension n, and both a and b are scalars. Eigenvalues* *of* $\mathbf{A}_n$ *are:* ab *with multiplicity* $n-1$*, and* $a(n+b)$ *with multiplicity one.*

Proof. Postponed in appendix. ■

Now it is possible to prove easily the following

Proposition 7 *The unique Nash of the Cobweb expectations game in case* $N \to \infty$ *is SREE if and only if* $|\beta| < 1$.

Proof. $\mathbf{A}_n = \mathbf{B}$ with $a = \beta/N$, $b = 0$ and $n = N$. From application of lemma 7 we know matrix $\mathbf{B}$ has N eigenvalues: 0 with multiplicity $N-1$ and β with multiplicity one. Therefore the ultimate condition for T to be a contracting map is $|\beta| < 1$. ■

A remark is worth do be done. As long as, agents reply to a state of the word instead that to others' expectations, their beliefs from the second order onward, are coordinated on the same point. That is, the i-th. component of $T^{\tau>2}\left(E^I_{t-1}p_t\right)$ is equal for each $i \in I$, and hence also to the τ-th. order individual belief on the aggregate expectation. Differently said, let be $E_{t-1,\tau-1}p_t \in \Im(0)_{\tau-1}$ the aggregation over agents' $\tau-1$-th. order beliefs, than

$$E_{t-1,\tau}p_t = \beta E_{t-1,\tau-1}p_t$$

for $\tau > 2$ is a surjective mapping of (1.17) according to (1.7). Therefore, the same dynamics can also be studied in terms of the unidimensional dynamics of beliefs on the aggregate expectation only, as in Guesnerie's original paper. Conditions for asymptotic convergence are more easily computed and result equal to the one found above ($|\beta| < 1$).

Nevertheless, the multidimensional dynamics triggered by iteration of T over the strategic space is a straight application of the classical IDSDS mechanism. Note that IDSDS triggers an updating dynamics in the whole strategic space and, only indirectly, in the space of the aggregate expectations. Moreover, the proposed framework enlightens how negligibility of individual agent's impact on aggregate quantities is what actually matters for the eductive argument in a perfect competition perspective. Hence, continuity of the agent set is not a necessary hypothesis. This is the starting observation for an easy generalization of eductive approach to a countable number of agents.

1.3.3 Generalization to a countable number of suppliers

Let consider now explicitly a countable and finite number of agents. We can discuss rationalizability of the Cobweb expectations game with respect a map $T_N : \Im(0)^N_{\tau-1} \to \Im(0)^N_{\tau}$ derived directly by (1.14) without using perfect competition approximation (1.15). T_N is formally given by

$$E^I_{t-1,\tau}p_t = \mathbf{B}_N E^I_{t-1,\tau-1}p_t \ , \tag{1.18}$$

where

$$\mathbf{B}_N \equiv \left(\frac{\beta_N}{1-\beta_N/N}\right)\frac{1}{N}\left(\mathbf{1}_N - I_N\right).$$

As in the previous section, $T_N\left(\Im(0)^N_{\tau-1}\right)$ represents the set of rationally justified strategies given the strategic set $\Im(0)^N_{\tau-1}$. Consistently $\Im(0)^N_\tau \equiv T^\tau_N\left(\Im(0)^N\right)$ is the set of all τ-order rationally justified agents' beliefs. In practice $\Im(0)^i_\tau$ is composed only by i's strategies that are best expectations function for at least one profile of strategies $E^{-i}_{t-1,\tau,1} \in \Im(0)^{-i}_{\tau-1}$.

Still, the Nash of the Cobweb expectations game (that is the REE of the Cournot model) is a SREE if and only if $\lim_{\tau\to\infty} T_N^\tau(\Im(0)) = \{0\}$. Therefore eductive stability of the system is given by conditions on eigenvalues of $\mathbf{B}_N$. As before, for T_N to be a contracting map its eigenvalues have to be inside the unit circle. The following is true.

Theorem 8 *The unique Nash of the Cobweb expectations game is SREE if and only if*

$$-\frac{N}{N-2} < \beta_N < 1\ , \tag{1.19}$$

for each $N \in \{2,\infty\} \subset \mathbb{N}$.

Proof. $\mathbf{A}_n = \mathbf{B}_N$ with $a = \beta_N/\left(N-\beta_N\right)$, $b = -1$ and $n = N$. From application of lemma 7 we know matrix $\mathbf{B}_N$ has N eigenvalues - $-\beta_N/\left(N-\beta_N\right)$ with multiplicity $N-1$ and $\beta_N\left(N-1\right)/\left(N-\beta_N\right)$ with multiplicity one. From the proposition just proved above we know that relevant inequalities are

$$\left|-\frac{\beta_N}{N-\beta_N}\right| < 1 \quad \text{and} \quad \left|\frac{\beta_N\left(N-1\right)}{N-\beta_N}\right| < 1\ ,$$

whose ultimate solution is

$$-\frac{N}{N-2} < \beta_N < 1\ ,$$

with $N > \beta_N$. This concludes the proof. ■

We have obtained eductive learning convergence depending on the structural parameter shaping both demand and supply curve, but on the number of firms also. In particular note that in the scenario of perfect competition ($N \to \infty$), (1.19) becomes $|\beta| < 1$ that is exactly the one found by Guesnerie (1992).

The finding has a straight intuitive meaning in that, in principle, more decisions units make coordination more demanding. Nevertheless this basic intuition exhibits a peculiar asymmetry. Specifically, supermodular complementarity between agents' expectations ($\beta_N > 1$) never allows eductive convergence, whereas, on the contrary, supermodular substitutability ($\beta_N < -1$) does allow expectations convergence conditional on proper restrictions on the maximum number of suppliers in the market.

As example take the model presented in section two. The relevant condition for eductive convergence in the model becomes

$$\frac{-N}{N-2} < -\frac{\alpha N}{\alpha + 2Nc} < 1 \ ,$$

or simply

$$\alpha < \frac{N}{N-3} 2c \ ,$$

since $\alpha > 0$ and $c > 0$. Note that in case $\alpha < 2c$ all the REE $P_t^*(N)$ are SREE independently from the number N of producers in the market. Differently, in case $\alpha > 2c$, there exists a value

$$N^* = \frac{3\alpha}{\alpha - 2c} \ ,$$

such that the unique REE is a SREE if and only if $N < N^*$.

The interpretation of the failure of rationalizability of the unique REE in the Cobweb model is that all possible values in $\Im(0)$ are a-priori possible because agents cannot rationally exclude prescriptions different from the REE one. In other words, firms are not any longer able to minimize the loss $\Lambda_{i,t-1}$ given by their stochastic forecast errors. In such a case, the market clearing price would become highly unpredictable given the intrinsic behavioural uncertainty that sums up to the exogenous stochasticity of the economy born by η_t. One can think such a situation would prevent firms from entering in the market even in presence of free entry and competitive cost structure of the industry. In the latter case no more than N^* firms would be willing to enter in the market. The thesis that expectations coordination mechanism could determine the failure of potentially competitive economies would strengthen the idea that communication policies impact on efficiency of real markets as well as, if not even more than, technological features. This issue is far beyond the scope of the present paper and is referred to future research. Here we just wanted to offer a new perspective on the economic relevance of eductive expectations coordination.

1.4 Conclusion

This paper has reshaped and generalized eductive learning argument to a class of games played by a countable set of agents. The argument relies on the observation that continuity of the agents' set is not necessary hypothesis to the eductive approach. To make the point in the context of a proper economic model, we introduced a simple Cournot model, the reduced form of which is Cobweb-like and its limit for an infinite number of agents replicates exactly the original Cobweb model. Nevertheless, the analysis can be equally referred to a generic Cobweb-like model with a countable set of agents. The main finding is that eductive learning convergence involves restrictions also on the number of active firms in the market as well as on the relative slope of demand and supply curves. Moreover, the classical perfect competition eductive learning conditions obtains in the limit of an infinite number of suppliers.

Finally, we gave just an insight on how eductive learning analysis of Cobweb-like models with a countable set of agents can rise new economically relevant issues. In general, barriers to entry can make markets with perfect competition cost structure eductively stable, whenever they would not naturally be.

Appendix

Details of the model

From (1.4) to (1.5). Knowledge of (1.1) and (1.2) implies[6]

$$E^i_{t-1}P_t = k - \alpha E^i_{t-1}S_{-i,t} - \alpha q_{i,t-i}$$

so that it is possible to recover a map from expected residual demand to expected price given by

$$E^i_{t-1}S_{-i,t} = \frac{k - \alpha q_{i,t-i} - E^i_{t-1}P_t}{\alpha}$$

and after some simple manipulation we obtain (1.5) substituting equation above in (1.4).

Marginal cost. Marginal cost is obtained as

$$MC_{i,t-1} = \frac{2Nc\,(k - d_{i,t-1})}{(N+1)\,\alpha + 2Nc} + d_{i,t-1}$$

so that, the individual mark-up μ_i on price is given by the difference between actual price and marginal cost. Formally

$$\mu_{i,t-1} = \frac{\alpha\,(k - d_{i,t-1}) - \alpha N\,(d_{i,t-1} - d_{N,t-1})}{(N+1)\,\alpha + 2Nc},$$

depends negatively on the slope of demand α and on both the absolute and relative level of the individual marginal linear cost indicated respectively by $d_{i,t-1}$ and $(d_{i,t-1} - d_{N,t-1})$. As usual, a more elastic demand curve damps market power. As normally required it is also true

$$\lim_{N\to\infty} \overline{P}_N = \lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N} MC_{i,t-i}$$

that is, price equals average marginal cost.

[6] We are assuming agents know η_t is a centred shock.

Proof. of lemma 7

By definition[7] eigenvalues of $\mathbf{A}_n$ are equal to eigenvalues of $a^{-1}\mathbf{A}_n \equiv (\mathbf{1}_n+b\mathbf{I}_n)$ multiplied by a. In turn eigenvalues of $a^{-1}\mathbf{A}_n$ are roots of $\det \mathbf{M}_n$ where $\mathbf{M}_n \equiv (\mathbf{1}_n + (b-\lambda)\ \mathbf{I}_n)$ is the generic $n \times n$ matrix whose entries are $(1+b-\lambda)$ on the main diagonal and one elsewhere. Our objective is to compute such roots.

Laplace rule for the calculation of the determinant is given by

$$\det \mathbf{M}_n = \sum_{i=1}^{n} (-1)^{i+j}\, m_{n,i,j} \det \mathbf{M}_{n,i,j}$$

where i and j stay respectively for row and column indexes, $m_{n,i,j}$ is the $(i,j)-$th. element of the matrix $\mathbf{M}_n$ and $\mathbf{M}_{n,i,j}$ is the $(n-1)\times(n-1)$ matrix obtained from $\mathbf{M}_n$ deleting the $i-$th. row and the $j-$th. column. Notice that for the matrix at hand the above formula simplify to

$$\det \mathbf{M}_n = (1+b-\lambda)\det \mathbf{M}_{n-1} - (n-1)\det \widetilde{\mathbf{M}}_{n-1}, \tag{1.20}$$

where $\widetilde{\mathbf{M}}_{n-1} \equiv \mathbf{M}_{n,2,j} = \mathbf{M}_{n,\widetilde{i},j,N}$ with $\widetilde{i} \geq 2$. This is evident from the fact that $\mathbf{M}_{n,\widetilde{i},j}$ can be obtained from $\mathbf{M}_{n,\widetilde{i}+1,j}$ just switching $\widetilde{i}-$th. column with $\widetilde{i}+1-$th. column, so that,

$$(-1)^{\widetilde{i}+j} \det \mathbf{M}_{n,\widetilde{i},j} = (-1)^{\widetilde{i}+1+j} \det \mathbf{M}_{n,\widetilde{i}+1,j}$$

is verified from straightforward application of determinant properties. Using Laplace rule to compute $\widetilde{\mathbf{M}}_{n-1}$ we have

$$\det \widetilde{\mathbf{M}}_n = \det \mathbf{M}_{n-1} - (N-1)\det \widetilde{\mathbf{M}}_{n-1}, \tag{1.21}$$

and, according to (1.20) and (1.21) we can state the following

$$\det \widetilde{\mathbf{M}}_n = \det \mathbf{M}_n + (\lambda - b)\det \mathbf{M}_{n-1},$$

and, finally,

$$\det \mathbf{M}_N = ((1+b-\lambda)-(n-1))\det \mathbf{M}_{n-1} - (\lambda-b)(n-1)\det \mathbf{M}_{n-2}. \tag{1.22}$$

that is, the algorithm recursively yielding the value of the generic determinant of $\mathbf{M}_n$.

[7] An eigenvalue λ of $\left(a^{-1}\mathbf{A}_n\right)$ is a scalar such that there exists a vector $\mathbf{x}$ satisfying $\left(a^{-1}\mathbf{A}_n\right)\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x}$ is called the eigenvector associated to λ. Therefore $a\lambda$ is the eigenvalue of A_n associated with the same eigenvector $\mathbf{x}$ since $\mathbf{A}_n\mathbf{x} = a\lambda\mathbf{x}$ also holds.

Initialization of (1.22) requires two data easily computed for $N = 2$ and $N = 3$, namely,

$$\begin{aligned} \det \mathbf{M}_2 &= (1 + b - \lambda)^2 - 1 = (b - \lambda)(b - \lambda + 2), \\ \det \mathbf{M}_3 &= (1 + b - \lambda)^3 - 3(1 + b - \lambda) + 2 = (b - \lambda)^2 (b - \lambda + 3). \end{aligned}$$

From inspection of the values of determinants so computed for $N = 4, 5, 6$...ect. we guess the general solution to be of the form

$$\det \mathbf{M}_n = (b - \lambda)^{n-1} (b - \lambda + n). \tag{1.23}$$

Let's verify the guess substituting (1.23) in (1.22). We have

$$\begin{aligned} \det \mathbf{M}_n = (b - \lambda)^{n-2} \left[((1 + b - \lambda) - (n - 1))(b - \lambda + n - 1) + (n - 1)(b - \lambda + n - 2) \right] = \\ = (b - \lambda)^{n-2} \left[(1 + b - \lambda)(b - \lambda + n - 1) - n + 1 \right] = (b - \lambda)^{n-1} (b - \lambda + n), \end{aligned}$$

that is (1.23) is the solution to (1.22) for each n.

Finally, roots of $\det M_n$ multiplied by a are: ab with multiplicity $n - 1$ and $a(b + n)$ with multiplicity one. Those are eigenvalues of $\mathbf{A}_n$.

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Chapter 2

Eductive Learning Matches Adaptive Learning

Abstract Eductive learning approach investigates rationalizability conditions of rational expectation equilibria. In case of Cobweb-like models, this paper shows how to modify the rule by which the rationalizable set is built in order to recover usual adaptive learning convergence restrictions. To the aim, the concept of bi-iterative rationalizable set is introduced.

2.1 Introduction

2.1.1 Learning approaches in Macroeconomics

According to a strict Microfounded interpretation, a rational expectations equilibrium (henceforth REE) is a Nash equilibrium of an appropriately defined coordination game (Evans, 1983). Normative meaning of Nash equilibrium is clear, but two basic questions in search of answer are beyond its descriptive interpretation. How agents acquire (the enormous amount of) information needed to coordinate on a particular equilibrium? And, whenever more than one equilibrium exists, which one has to be considered the relevant one?

The introduction of learning in macroeconomics is a way to deal with epistemic foundations and multiplicity issue maintaining a Microfounded and decentralized point of view. The two main learning approaches in Macroeconomics are eductive and adaptive learning. Eductive learning (Guesnerie, 1992, 2005) investigates conditions under which common knowledge of rationality and of the model underlying the economy are enough to make fully rational agents to rationally exclude any doubt that someone will fail to conform to REE prescriptions. That is, eductive learning spells out conditions under which a particular REE coincides with the rationalizable set of the coordination game entailed by the self referential-nature of the economy. The updating of beliefs, triggered by the rationalizability recursive mechanism, is interpreted as a form of learning in notional time. Eductive convergence conditions depend on the relative slope of demand and supply curve, but also on the number of firms active in the market as well (Gaballo 2009).

Adaptive learning (Marcet and Sargent, 1989; Evans and Honkapohja, 2001) questions the coordination on REE from an evolutive perspective. This approach assesses whether or not REE can be achieved in real time when agents form expectations using linear recursive regressions on observable exogenous variables as normally economists do in reality. Agents using adaptive learning are assumed to hold a particular form of boundness in that they estimate the economy as it is unaffected by their forecast. In practice, they use a misspecified structural equation for their estimates that encompasses the rational expectation equilibrium form. This kind of misspecification vanishes asymptotically as learning dynamics successfully converges. Nevertheless, such peculiar departure from full rationality might cast doubts on the robustness of results with respect different deviations from full cognitive capacities.

2.1.2 A bridge in between

Eductive and adaptive learning approaches have completely different methodologies. The latter concerns only learning about exogenous determinants and by-passes all issues linked to others' expectations, whereas the former focuses exclusively on the behavioral uncertainty problem entailed by coordination on REE. As one expects they give different results. For example, with reference to the Cobweb model, conditions for eductive convergence is $|\beta| < 1$ whereas conditions for adaptive convergence is $\beta < 1$, where β measures the feedback impact of aggregate expectation on output.

Evans (1985), Evans and Guesnerie (1993), Evans and Honkapohja (2001) and Evans (2001) provide and discuss the concept of iterative expectational stability as bridge between the two approaches. Iterative expectational stability concept relates to adaptive learning framework in that the property of iterative expectational stability refers to the contracting nature of the discrete iteration of the T-map, the coefficients mapping from perceived law of motion to the actual one. The iterative expectational convergence conditions results to be the same of eductive stability ones provides the cobweb model as a sufficient homogeneous structure. Nevertheless, iterative expectational stability substantially deviates from the idea of agents acting as econometricians.

The purpose of this paper is to establish a further point of contact between adaptive learning and eductive learning according to the eductive perspective. The scope is to identify at least a sufficient restriction or modification of the iterative rule governing the iterated deletion of strongly dominated strategies, henceforth IDSDS, to make eductive convergence conditions match adaptive ones. Differently said, we want to understand whether or not adaptive learning conditions can be recovered also with the same epistemic assumptions of eductive learning.

The notion of rationalizable set is substituted with a different ad-hoc iterative solution concept tagged *bi-iterative rationalizable set* whose recursive construction rule is obtained from IDSDS, the classical one of the rationalizable set, adding a further dimension to the iteration. Therefore, the same epistemic assumptions sustaining the rationalizable set are maintained, nevertheless they converges to the unique REE at different conditions. This is because, the classical rationalizable set is built up updating iteratively the whole strategic space in notional time, differently, the bi-iterative rationalizable set is the result of iterative updating of the strategic space both in notional time *and* by agent. In particular, bi-iterative expectational convergence conditions mimic adaptive learning ones and does not depend on the number of firms operating in the market.

2.2 Cobweb-like games with a countable set of agents

2.2.1 Basic Framework

Consider a simple Cobweb-like one shot market where price is determined linearly by the aggregate expectation over a countable set of agents $I = \{1, 2, .., i, .., N\}$. The economy is represented by

$$P = \alpha + \beta \, E(P) \tag{2.1a}$$

$$E(P) \equiv \frac{1}{N} \sum_{i \in I} E^i(P), \tag{2.1b}$$

where P is price, α and β are exogenous parameters and $E^i_{t-1}P_t$ is individual expectation of agent $i \in I$. A such economy has been recovered from a simple Cournot model in Gaballo (2009) to which the interested reader is referred for an example of Microfundation. Here we will abstract from real time and stochasticity normally included in Cobweb-like models to simplify the notation. Such features do not play any role in the eductive argument. The basic framework can easily be extended without changing conclusions. The following assumption concerns what agents are supposed to know about exogenous determinants of the economy.

Epistemic assumption on the state of the word The working of the economy entailed by (2.1) is common knowledge among agents.

Common knowledge (Aumann, 1974) about the working of the economy means that agents know that every one knows that every one knows at infinitum that every one knows (2.1). Therefore, nobody doubts that everybody has knowledge of the model.

Knowledge of the model is enough to make agents to compute the fundamental price, that only depends on the structure of the economy perfectly known as state of the word. The latter is obtained as $\overline{P} = \alpha/\left(1-\beta\right)$ in case aggregate expectation is correct. Consistently, it is possible to rewrite (2.1) in terms of deviations of actual price from the fundamental one according to

$$p = \beta\, e(p) \tag{2.2a}$$

$$e(p) \equiv \frac{1}{N}\sum_{i\in I} e^{i}\left(p\right) \tag{2.2b}$$

where $p \equiv P-\overline{P}$ is price deviation of actual price from the fundamental one $e(p)$ and $e^{i}\left(p\right)$, from here onward simply e and e^{i}, are respectively aggregate and individual expected price deviation p. From (2.2) notice that the actual price is equal to the fundamental price if and only if every agent expects the fundamental price. Nevertheless epistemic assumption on the state of the word is not sufficient to foresee actual price because deviations from the fundamental price can occur as agents do not expect the fundamental price.

At this point, we have to specify how behavioral data are generated, namely how agents form their expectations. Assume each agent is rational and has incentive to consistently forecast actual price. Specifically, each one has to minimize the following individual loss function

$$\Lambda_i = \left|p - e^{i}\right| \tag{2.3}$$

where p moves according to (2.2). Rationality is still not sufficient to exclude beliefs that others would not forecast the fundamental price. Finally, the following assumption strengthens and completes the epistemic structure in the model.

Epistemic assumptions on behavioral variables The following facts are common knowledge among agents: i) agents are rational, ii) each individual expectation e^{i} belongs to a centred closed interval $\Im_i$ of the fundamental price $\overline{P}$.

A corollary of the hypothesis above is that the coordination game defined as follows is common knowledge among agents.

Definition 9 *A Cobweb expectations game is a triple* $\Gamma_N \equiv \{I, \Im^N, \{\Lambda_i\}_I\}$ *composed by a countable set of agents* $I \equiv \{1, 2, .., i, .., N\}$*, a strategic space* $\Im^N \equiv \prod_{i=1}^{N} \Im_i$*, and a family of loss functions, one for each* $i \in I$*,* $\Lambda_i : \Im(0)^N \to \mathbb{R}$ *defined by (2.3).*

The strategic space of the Cobweb expectations game is expressed directly in terms of expectations. In fact, one can think there always exists a one to one relation between individual actions and individual expectations on actual price. Individual best expectations functions[1] for the game Γ_N, namely e^{i*}, are given by

$$e^{i*} = \frac{\beta}{N-\beta} \sum_{j \in I - \{i\}} e^j. \tag{2.4}$$

A Nash equilibrium (the REE) obtains in correspondence of $e^{i*} = 0$, $\forall i \in I$. The basic question eductive learning literature answers is: do common knowledge of the model and of rationality guarantee agents' beliefs convergence to the unique (or a particular) REE? Next subsection will briefly discuss results from the standard analysis.

2.2.2 Rationalizable set and eductive learning

Eductive learning assumes agents behave like economic theorists. Hence they deduce according to game-theory reasoning the rational(izable) expectation each agent should maintain given common knowledge of rationality and of the game. The usual reference point for such problem is the iterative solution concept of rationalizable set introduced independently by Bernheim (1984) and Pearce (1984). They provided a constructive way to define it through iterated deletion of strongly dominated strategies that we briefly summarize and comment below.

Let index by τ notional time, that is, the iteration of the deletion mechanism. Consider the game Γ_N, the precise rule of the IDSDS mechanism is given by

$$\begin{aligned} \Im_{i,\tau} &= \Im_{i,\tau-1} - \{e^i \in \Im_{i,\tau-1} : \forall e^{-i} \in \Im_{-i,\tau-1}, \\ &\exists\, \widehat{e}^i \in \Im_{i,\tau-1} : \Lambda_i(\widehat{e}^i, e^{-i}) < \Lambda_i(e^i, e^{-i})\}, \quad \forall i \in I \end{aligned} \tag{2.5}$$

where $\Im_{-i} \equiv \prod_{j \in I - \{i\}} \Im_j$. Wordily, $\Im_{i,\tau}$ differs from $\Im_{i,\tau-1}$ as all individual strategies e^i that are not best reply for at least one strategies profile belonging to the $\Im_{-i,\tau-1}$ belongs to the latter but not to the former.

Definition 10 *The rationalizable set $\Re(\Gamma_N) \subseteq \Im^N$ is the set obtained as*

$$\Re(\Gamma_N) \equiv \prod_{i \in I} \left(\bigcap_{\tau=0}^{\infty} \Im_{i,\tau} \right).$$

[1] Form here onward we label e_i^* the best expectation function of agent i and e_i just a point belonging to agent i's expectation space.

Notice that iteration in notion time apply to all individual strategy sets equally, so there is not particular order by which individual strategic sets are considered as long as at each step iteration is completed for each agent ($\forall i \in I$ in (2.5)). The rationalizable set is the finest refinement of the strategic set justified by common knowledge of individual rationality and of the game. The solution concept of Nash equilibrium corresponds to the rationalizable set if and only if the latter is a singleton.

The eductive argument is based on the concept of strongly rational expectations equilibrium (SREE). Below we report the definition of SREE given by Guesnerie (1992).

Definition 11 *A strongly rational-expectations equilibrium (SREE) is a rational expectations equilibrium that is the unique rationalizable equilibrium of the Cobweb expectations game.*

Evans and Guesnerie (1993) use with the same meaning of SREE the expression "eductively stable equilibrium" or "stable under eductive learning". Each rationalizable REE is stable in the sense that beliefs of others will deviate from the REE prescriptions will be rationally excluded at same point of the IDSDS. Hence, given common knowledge of rationality and of the game, a REE is eductively stable if and only if all individual expectations will converge (in notional time) to REE independently on initially believed displacements from it. In other words, the existence of a SREE can justify convergence of players on this particular outcome through a process of mental reasoning as if they were game theoretic economists knowing all the objective features of the problem. Eductive learning literature investigates conditions under which the requirement that agents' beliefs in the Cobweb expectations game belong to the rationalizable set is enough to pin down a unique strategy profile, namely the unique REE of the Cobweb expectations game. More formally said, the aim is to spell out conditions under which $R(\Gamma_N) = \{0\}$. Although the rationalizable set can also differ from the unique REE for some (α, β) specification of Γ_N. In such a case (2.5) is not able to refine the strategic set and hence it leave expectations unrestricted. Gaballo (2009) provides an analysis of Cobweb-like games implementing explicitly (2.5). The results are summarized by the following theorem.

Theorem 12 (Gaballo 2009) *Given an expectations Cobweb game Γ_N, then $\Re(\Gamma_N) = \{0\}$ if and only if $\frac{N}{N-2} < \beta < 1$, whereas $\Re(\Gamma_N) = \{\Im^N\}$ otherwise.*

Nevertheless, even if the concept of rationalizable set is sustained by common knowledge of rationality and the game, the particular iteration by which it is built up does not spring straightway from epistemic assumptions that sustain it. In other words, recursion (2.5) is just a way to use common knowledge to iteratively delete strongly dominated strategies, no epistemic assumption implies that particular order. In the following, we propose a different recursion that equally exploits the same epistemic assumptions, but follows a different deleting order. We will prove this rule yields the same rationalizable set outcomes, but at conditions different from the classical ones. This case is particularly interesting because such convergence conditions result to be the same ones required by adaptive learning convergence.

2.3 Eductive learning according to *bi*-IDSDS

2.3.1 *Bi*-IDSDS mechanism

In this section we will introduce the concept of *bi*-IDSDS, an iterative solution concept obtained by a concise modification of the classical IDSDS rule. The idea beyond the modification is really simple: instead of running IDSDS for the whole strategic set at each τ, as the classical rule provides for, *bi*-IDSDS does exactly the same but just for a subset of the strategic set that is iteratively updated (this implies a second iteration).

Consider the game Γ_N, the recursive rule of the *bi*-IDSDS mechanism is formally given by

$$\begin{aligned}\Im_{i,k-i,\tau} = \Im_{i,k-i,\tau-1} - \{e^i \in \Im_{i,k-i,\tau-1} : \forall e^{-i} \in \Im_{-i,k,\tau-1}, \\ \exists\ \widehat{e}^i \in \Im_{i,k-i,\tau-1} : \Lambda_i(\widehat{e}^i, e^{-i}) < \Lambda_i(e^i, e^{-i})\}, \quad \forall i \in I_k\end{aligned} \tag{2.6a}$$

where

$$\Im_{-i,k} \equiv \prod\nolimits_{j \in I_k - \{i\}} \Im_{j,k-j} \tag{2.6b}$$

and

$$\Im_{i,k-i} \equiv \lim_{\tau \to \infty} \Im_{i,k-1-i,\tau} \quad \text{with} \quad \Im_{i,k-i} \equiv \Im_i \quad \text{if} \quad k-i<1 \tag{2.6c}$$

for $I_k \equiv \{1, 2, .., k\}$ and k going from 1 to N.

Let's analyse each component of the recursion to focus on differences. Algorithm (2.6a) has the same structure of (2.5), but it differs because : *i)* strategic sets considered ($\Im_{i,k-i}$) are labeled with an additional index $k-i$ denoting a second recursion and *ii)* it is implemented for each i belonging to I_k and not to I, the former being a subset of the latter. Keep in mind, k is the index of the second recursion going from 1 to N and i is the i-th agent. Definition (2.6c) explains how to build up the new object $\Im_{i,k-i}$. The latter differs from $\Im_i$ just in case $k-i<1$. In such a case $\Im_{i,k-i}$ is a strict subset of $\Im_i$ obtained after infinite recursive application of (2.6a) at the stage $k-1$.

In practice, for $k=1$ only agent 1's strategic set, namely $\Im_{1,0} = \Im_1$ is considered for deletion of strongly dominated strategies since $I_k \equiv \{1\}$. In such trivial case iteration is redundant as $\Im_{-i,1}$ is τ-invariant because no agents are in $I_k - \{i\}$. For $k=2$ both

$\Im_{1,1} = \lim_{\tau\to\infty} \Im_{1,0,\tau}$ and $\Im_{2,0} = \Im_2$ are considered for updating as $I_k \equiv \{1,2\}$. Notice that $\Im_{1,1}$ is both the resulting agent 1's strategic space after τ-iteration at $k = 1$ and the starting agent 1's strategic space for $\tau-$iteration at $k = 2$. Consistently, at a certain stage k, $\Im_{1,k-1}$, $\Im_{2,k-2}, \ldots$, $\Im_{k-1,1}$ are the resulting strategic spaces of agents belonging to I_{k-1} after $\tau-$iteration at infinity, but also the starting strategic spaces of agents belonging to I_k for $\tau-$iteration at the following $k+1$ stage. In sum, (2.6) provides for N recursive application of (2.5) appropriately nested. Finally *bi*-iterative rationalizable set is obtained by intersection of all strategic sets for each k and τ.

Definition 13 *Bi-iterative rationalizable set* $\widetilde{\Re}(\Gamma_N) \subseteq \Im^N$ *is the set obtained as*

$$\widetilde{\Re}(\Gamma_N) \equiv \prod_{i\in I} \left(\bigcap_{k=1}^{N} \bigcap_{\tau=0}^{\infty} \Im_{i,k-i,\tau} \right).$$

It is not a concern of this work to assess all general properties of *bi*-iterative rationalizable set. For the scope of the paper it is just enough to prove an equivalence with the rationalizable set in the context of Cobweb-like coordination games. Next section presents the main theorem.

2.3.2 *Bi*-iterative eductive convergence conditions

In this section we will prove that the *bi*-iterative rationalizable set coincides with the unique REE at the same conditions of adaptive learning convergence. The more straight intuition for the result is geometric. The classical IDSDS shapes a N-dimensional contracting mapping whose domain is the whole strategic space. Differently, as long as iterative deletion of strongly dominated strategies is implemented iteratively (from here the term '*bi*-iterative') for each agent at time, the strategic space is contracted one dimension at time. This mechanism cuts all cross-individuals effects. Therefore each iterative deletion at stage k works orthogonally with respect to all individual strategic space of agents $i > k$ in that all correspondant $\Im_{i,k-i}$ are simply not updated according to (2.6c).

Theorem 14 *Given an expectations Cobweb game* Γ_N, *then* $\widetilde{\Re}(\Gamma_N) = \{0\}$ *if and only if* $\beta < 1$, *whereas* $\widetilde{\Re}(\Gamma_N) = \{\Im^N\}$ *otherwise.*

Proof. The strategy for the proof. is iterative. We will build up the iteration for $k = 2$ ($k = 1$ is trivial) and then for a generic k in case of convergence. Finally we will discuss conditions for convergence.

First step: the iteration for $k = 2$. Let be $e^i_{k-i,\tau}$ the generic element of the strategic set $\Im_{i,k-i,\tau}$. For $k = 2$ the *bi*-IDSDS updating mechanism concerns only agents

1 and 2 and it is simply represented by

$$\mathbf{e}_{2,\tau+1} = \mathbf{a}_2 + \mathbf{B}_2\ \mathbf{e}_{2,\tau} \tag{2.7}$$

with

$$\mathbf{e}_{2,\tau} \equiv \begin{bmatrix} e^1_{1,\tau} \\ e^2_{0,\tau} \end{bmatrix},\ \mathbf{B}_2 \equiv \begin{bmatrix} 0 & \frac{\beta}{N-\beta} \\ \frac{\beta}{N-\beta} & 0 \end{bmatrix} \text{ and } \mathbf{a}_2 \equiv \begin{bmatrix} \frac{\beta}{N-\beta} \\ \frac{\beta}{N-\beta} \end{bmatrix} \sum_{j=3}^N e^j.$$

The mechanism is explained as follows. Both $\mathbf{B}_2$ and $\mathbf{a}_2$ are derived directly from best reply function (2.4). $\Im_{1,1,\tau+1}$ and $\Im_{2,0,\tau+1}$ are respectively sets composed by all the expectations $e^1_{1,\tau+1}$ and $e^2_{0,\tau+1}$ that are best expectations for at least a profile of strategies belonging to the space $\Im_{1,1,\tau} \times \Im_{2,0,\tau} \times \prod_{j=3}^N \Im_j$, namely $e^1_{1,\tau} \times e^2_{0,\tau} \times \prod_{j=3}^N e^j$. In other words, (2.7) entails a bidimensional mapping in τ going from $\Im_{1,1,\tau} \times \Im_{2,0,\tau}$ to $\Im_{1,1,\tau+1} \times \Im_{2,0,\tau+1}$ and preserving $\prod_{j=3}^N \Im_j$. Therefore, at this stage, only individual strategic spaces of agents belonging to $I_k = \{1, 2\}$ are updated according to $\tau-$iteration. In particular, $\tau-$convergence to a singleton in the (expectational) strategic space of agents 1 and 2, namely

$$\lim_{\tau\to\infty} e^1_{1,\tau} = \lim_{\tau\to\infty} e^2_{0,\tau} = \frac{\beta}{2(1-\beta)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sum_{j=3}^N e^j \tag{2.8}$$

requires $|\beta/(N-\beta)| < 1$ that is $\beta < N/2$. In such a case, according to (2.6a), $\Im_{1,2} = \Im_{2,1}$ for $k = 3$.

Note also that for $N = 2$, it is $\mathbf{a}_2 = [0\ 0]'$ and so both $e^1_{1,\infty} = e^2_{0,\infty} = 0$, that is $\widetilde{\Re}(\Gamma_2) = \{0\}$, if and only if $\beta < 1$. For $N > 2$ we have to consider further iteration in k.

Second step: the iteration for a generic k. So, let's now run the generic k-th. step as

$$\mathbf{e}_{k,\tau+1} = \mathbf{a}_k + \mathbf{B}_k\ \mathbf{e}_{k,\tau} \tag{2.9}$$

with

$$\mathbf{e}_{k,\tau} \equiv \begin{bmatrix} e^1_{k-1,\tau} \\ e^k_{0,\tau} \end{bmatrix},\ \mathbf{B}_k \equiv \begin{bmatrix} 0 & \frac{\beta}{N-(k-1)\beta} \\ \frac{\beta(k-1)}{N-\beta} & 0 \end{bmatrix} \text{ and } \mathbf{a}_k \equiv \begin{bmatrix} \frac{\beta}{N-(k-1)\beta} \\ \frac{\beta}{N-\beta} \end{bmatrix} \sum_{j=k+1}^N e^j$$

being guessed

$$e^1_{k-1,\tau} = e^i_{k-i,\tau} \text{ for } i < k \text{ and } \forall\tau \tag{2.10}$$

as result of successful $\tau-$iteration at $k-1$ to a singleton in the strategic (expectational) space composed by the union of all individual strategic space of agents belonging to I_{k-1} (correspondingly to (2.8)). Still both $\mathbf{B}_k$ and $\mathbf{a}_k$ are derived directly from best reply functions (2.4) taking properly in to account the guess (2.10). In this respect (2.9) is still a bidimensional mapping in τ going from $\Im_{1,k-1,\tau} \times \Im_{k,0,\tau}$ to $\Im_{1,k-1,\tau+1} \times \Im_{k,0,\tau+1}$ preserving

$\prod_{j=k}^{N} \Im_j$. In fact, according to (2.10), the strategic spaces $\Im_{i,k-i,\tau}$ for $1 > i > k$ follow identical updating of $\Im_{1,k-1,\tau}$.

Condition (2.10) has been proven to be true for $k = 3$ as we explicitly computed rationalizable strategies at $k = 2$. Let us check that this is the case for all consecutive steps in k, that is, let us prove that given (2.10) we have

$$\lim_{\tau\to\infty} e^1_{k-1,\tau} = \lim_{\tau\to\infty} e^k_{0,\tau} \tag{2.11}$$

in case convergence obtains, so that (2.10) is confirmed at $k+1$. Let compute explicitly $\lim_{\tau\to\infty} \mathbf{e}_{\tau,k}$

$$\lim_{\tau\to\infty} \mathbf{e}_{k,\tau} = (1+\mathbf{B}_k)^{-1} \mathbf{a}_k =$$

$$= \left(1 - \frac{\beta^2 (k-1)}{(N-\beta)(N-(k-1)\beta)}\right)^{-1} \begin{bmatrix} 1 & \frac{\beta}{N-(k-1)\beta} \\ \frac{\beta(k-1)}{N-\beta} & 1 \end{bmatrix} \begin{bmatrix} \frac{\beta}{N-(k-1)\beta} \\ \frac{\beta}{N-\beta} \end{bmatrix} \sum_{j=k+1}^{N} e^j =$$

$$= \frac{(N-\beta)(N-(k-1)\beta)}{N(N-\beta k)} \begin{bmatrix} \frac{\beta}{N-(k-1)\beta} + \frac{\beta^2}{(N-\beta)(N-(k-1)\beta)} \\ \frac{\beta^2(k-1)}{(N-\beta)(N-(k-1)\beta)} + \frac{\beta}{N-\beta} \end{bmatrix} \sum_{j=k+1}^{N} e^j =$$

$$= \frac{\beta}{N-\beta k} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sum_{j=k+1}^{N} e^j,$$

hence (2.11) is proved. Specifically, $\lim_{\tau\to\infty} e^i_{k-i,\tau} = e^i_{k+1-i,\tau}$ for $i < k+1$ are independent each other and they only reply to profiles of strategies of players belonging to $I - I_{k+1}$, Therefore (2.11) is true at each τ. Moreover, notice that the common reply coefficient $\beta/(N-\beta k)$ calculated above is consistent with the iteration of (2.9). Finally for $k = N$, in case of convergence, we get $e^i_{N-i,\infty} = 0$ for each $i \in I_N = I$ (that is $\widetilde{\Re}(\Gamma_N) = \{0\}$) as $\sum_{j=N+1}^{N} e^j = 0$.

Last step: convergence conditions. Now, let's investigate conditions for convergence. That is, we look for the set of β and N such that $\mathbf{B}_k$ is a contracting map. It occurs whenever eigenvalues of $\mathbf{B}_k$ lie inside the unit circle, for each k going from 2 to N, that is whenever

$$\left| \frac{\beta^2 (k-1)}{(N-(k-1)\beta)(N-\beta)} \right| < 1. \tag{2.12}$$

Since $\beta^2 (k-1)$ is always positive we are going to consider two different cases for $(N-\beta(k-1))(N-\beta)$ being respectively positive and negative.

a) First case: $(N-\beta(k-1))(N-\beta) > 0$ occurs whenever

$$\beta < \frac{N}{k-1} \quad \text{or} \quad \beta > N. \tag{2.13}$$

It follows that (2.12) holds for

$$\beta < \frac{N}{k}. \tag{2.14}$$

If this is the case $\forall\ k \in \{2, 3, 4...N\} \subset \mathbb{N}$, it is easy to prove $\beta < 1$ (obtained for $k \to N$) to be the most restrictive condition.

b) Second case: $(N - \beta(k-1))(N - \beta) < 0$ occurs whenever

$$\frac{N}{k-1} < \beta < N. \tag{2.15}$$

It follows that (2.12) holds for $2(k-1)\beta^2 - kN\beta + N^2 < 0$ whose solution is

$$\frac{k - \sqrt{k^2 - 8(k-1)}}{4(k-1)} N < \beta < \frac{k + \sqrt{k^2 - 8(k-1)}}{4(k-1)} N, \quad \text{if} \quad k > 4 + 2\sqrt{2} \tag{2.16}$$

and *never* otherwise. Given that

$$\frac{k - \sqrt{k^2 - 8(k-1)}}{4(k-1)} > 1$$

is always true for feasible k values, it follows that (2.15) and (2.16) have no intersection for any feaseable k value. Therefore, this case is never of interest.

We conclude $\beta < 1$ to be the only relevant condition for convergence in all cobweb expectations game for whatever $N > 2$.

Finally notice that in case convergence fails in one of the k step, then the strategic space is left unrestricted since it is not possible to uniquely define the *bi*-iterative rationalizable set, because it would be not agents'order-independent. This concludes the proof. ■

2.4 Conclusion

This paper has proved that there exist at least one modification of the IDSDS rule yielding an iterative solution different from the rationalizable set that, in Cobweb like models, can select the REE as a singleton at the same conditions required for adaptive learning conditions. We have shown that adaptive learning conditions can be recovered also with the same epistemic assumptions of eductive learning: common knowledge of rationality and of the model and *full* cognitive capacity. In other words, the convergence requirements identified by adaptive learning approach are not unavoidably linked to the bounded rationality hypothesis. Moreover, even if agents maintain all cognitive capacities, and hold common knowledge of rationality and of the model, there is not a unique way to recover the REE exploiting these epistemic assumptions. Hence, eductive learning conditions are dependent on the kind of iterative solution concept that is used, as adaptive learning approach is dependent from the particular form of bounded rationality assumed.

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Part II

Adaptive Learning

Chapter 3

Assessing others' rationality in real time

Abstract This paper shows how to make adaptive learning schemes work in a truly behavioral uncertainty problem. We consider a simple self-referential cobweb model where two institutional forecasters polarize private sector expectations. Unlike the standard adaptive learning approach, agents are not boundedly rational in that they acknowledge their own non-negligible influence on output and they are perfectly informed about fundamentals of the economy. Nevertheless, no common knowledge is assumed and institutional forecasters can only have noisy perceptions of other's simultaneous expectations. This minimal deviation from full knowledge gives rise to adaptive learning about the other's rationality as optimal solution to their own forecast problem. The decentralized learning system possibly converges to: (a) the unique rational expectation equilibrium (REE) or (b) excess volatility equilibria tagged behavioral sunspot equilibria (BSE). The latter may arise as a coordination failure because learning dynamics can generate the self-fulfilling reciprocal belief that the other agent is irrationally exuberant.

3.1 Introduction

3.1.1 Expectations and adaptive learning

This paper concerns epistemic foundations of rational expectation equilibrium (REE). REE is the natural equilibrium concept of long-run macroeconomic analysis. Its plausibility has been criticized from the theoretical and empirical points of view. Here we are concerned with theoretical justifications of REE relevancy. The aim is a better appraisal of conditions under which agents using adaptive learning schemes are able to coordinate on REE. This line of research has been substantially developed in the last two decades (Marcet and Sargent, 1989 a,b; Evans and Honakpohja, 2001). The key idea is to model agents acting as econometricians in self-referential models. Agents form expectations in real time using an estimated model of the economy. In particular, they recursively regress exogenous variables on the variable of interest as if the economy was independent of their forecasts. Agents' behavior is generally presented as the result of a kind of bounded rationality, in that agents miss the self-referential nature of the economy.

Adaptive approaches to expectations generally improve empirical performances of many macroeconomic models and also constitutes a theoretical selection device in the presence of equilibria multiplicity. Notwithstanding, the bounded nature of agents' rationality raises the issue of whether and how much the results rely on the particular form of boundedness assumed. In particular, the hypothesis that agents believe what they are learning about to be independent of their beliefs does not account for interdependence and co-movements of individual expectations usually argued to be responsible for crisis triggering or excess volatility phenomena. This paper sets up a basic framework to let learning schemes in real time deal with such issues overcoming the limits of the bounded rationality hypothesis. How adaptive learners with non negligible impact on aggregate expectation can independently solve (or fail to solve) the truly behavioral uncertainty problem embodied by REE definition is investigated.

3.1.2 Interdependency in expectations

In real economies, the presence of institutional forecasters acting as focal points for private expectations provides a natural example of the economic relevance of the problem at hand. Let us consider the forecasting activity of a big actor, like Apple, in the software market. Even if Apple has its own theory about the course of the market, it probably cannot neglect the influence of expectations of other big actors such as Microsoft, on the aggregate expectation. An announcement by Microsoft of a downward expectation about future software market profits will generally affect the ocean of small operators in the software market, who will probably revise their expectations downward. Given the self-referential nature of the economy, Apple has to consider this fact in order to hold consistent expectations. In turn, if Microsoft recognizes the influence of Apple on aggregate expectations, it has to do the same with respect Apple' expectations. The reader can easily think up other examples of the same strategic motives for interaction of expectations among rating agencies, fiscal authorities, central banks, financial institutes and so on. Agents with a non negligible effect on aggregate expectations play a coordination game dictated by self-referential economies. In this game holding rational expectations is not the best expectation (in the mean square error sense) if others do not simultaneously do the same. Non trivial indeterminate expectations interdependency therefore arises in the form of forecasting the forecast of others.

In the profession, it is normal practice to presume coordination of beliefs assuming that common knowledge holds to some extent. Common knowledge implies that agents have no doubts about what others know, so that each agent trivially maintains consistent beliefs about what others are doing. Coming back to our example, Apple's forecasting problem would be bypassed by the assumption of common knowledge that Apple and Microsoft both know the right economic theory, are rational and look at the same data. Nevertheless, as the limited number of papers on this topic demonstrates, the problem of endogenous emergence of belief consistency based on the concept of common knowledge is highly problematic. Firstly, modelling dynamic convergence in the infinite-dimensional space of higher order beliefs is a titanic effort. More essentially, the concept of common knowledge is not strictly required for emergence of a Nash (for details see to Aumann and Brandenburger 1995). This paper copes with convergence of agents' beliefs on REE in real time without assuming any common knowledge. A first task is to understand how a state such as that entailed by assuming common knowledge can arise from a prior disequilibrium when agents learn adaptively about each other's rationality without sharing any knowledge. More specifically, how can coordination of expectations required for REE to emerge arise endogenously from a decentralized learning mechanism?

3.1.3 Decentralized learning about others' rationality

In principle, one might statistically assess the rationality of others' expectations as if they were just exogenous data, provided the problem can be correctly identified. Using this idea we set up an extremely simple laboratory exercise. We consider a stylized first order self-referential market where a certain exogenous process and the aggregate expectation on the current output gap jointly determine the actual output gap. The problem of "forecasting the forecast of others" exists because two institutional forecasters polarize public expectations; both have non negligible impact on the aggregate, hence both have incentive to care about eventual displacements of the other's expectation from the rational expectation. To isolate the behavioral uncertainty issue, we assume institutional forecasters have perfect knowledge of the exogenous determinants of the model underlying the economy, but they only have a noisy signal of others' simultaneous expectations. The agents look at each other's expectations because they are uncertain about the process governing them. In order to improve the precision of their own forecasts, both estimate a coefficient weighting the noisy signal about the other's expectation in the expectation formation process as it would provide information on an unknown exogenous stochastic factor displacing aggregate expectation from the rational one. A REE occurs whenever this coefficient is zero so that only relevant, truly exogenous information is used, that is, expectations are formed independently.

Nevertheless, convergence to REE is not trivially guaranteed. As long as the consistency of beliefs is generated endogenously, stochastic departures from REE can potentially produce new equilibria. This happens because the system is not bounded by any epistemic condition imposed a-priori by common knowledge and could be stuck in a coordination failure trap. Intuitively, if all relevant exogenous information is available and agents are rational[1], departures of expectations from the rational one occur because expectations are not formed independently in the sense described above. In such cases, noisy information about the simultaneous expectation of the other "big actor" in the market is partially informative about displacements of the aggregate expectation from the rational one. Such departures are not correlated with any economically relevant exogenous variable of the model, but are only due to observational errors. Both agents have therefore incentive to maintain their expectations conditioned to the noisy signal because, in this situation, it becomes a predictor of actual output gap dynamics. This incentive could be strong enough to make the interdependency of expectations a self-fulfilling prophecy. A second task of this work is to investigate the existence of learneable equilibria entailing such interdependence of expectations.

The paper basically shows two results. Firstly, the unique REE arises from a prior disequilibrium in a large region of coefficient space without relying on any common knowledge property. It results from decentralized learning dynamics in which every agent assess rationality of the other in real time. Moreover, attainability of REE is proven to be particularly robust to correlation and systematic bias in observational errors. Secondly, *even if* institutional forecasters are perfectly informed about economic fundamentals and acknowledge their own influence on the economy, the existence of at least one learnable equilibrium different from REE (which I tagged behavioural sunspot equilibrium, BSE from here onward) is analytically proven. This happens because in some cases agents' observational errors can generate the self-sustaining belief of the other being irrationally exuberant. These equilibria exhibit excess volatility around the rational expectation, determined by noisy signals of institutional forecasters' expectations persistently entering the equilibrium solution. In contrast to other sunspot equilibria and consistently with the general approach, BSEs do not imply any common knowledge property, but arise from endogenous coordination among non-cooperative agents. No qualitative features of the results rely on special assumptions about the type of distribution generating observational errors.

[1]But they do not know whether or not others are rational!

3.1.4 Related literature

Recent literature tackles the issue of structural heterogeneity in adaptive learning models (Berardi, 2007; Branch and Evans, 2006; Giannitsarou, 2003; Evans, Honkapohja and Marimon, 2001; Guse, 2005; Honkapohja and Mitra, 2006) but the problem of interaction among learners is not considered. Granato, Guse and Wong (2007) seems to be the only study investigating the case of minimal social interaction between two types of agents using adaptive learning.[2] Nevertheless, in all these papers the existence of heterogeneity does not matter since bounded rationality hypothesis is assumed. Since agents do not recognize the self-referential nature of the economy, the behavioral uncertainty problem is never relevant. Here we build a more general perspective on the issue of learning interactions, making adaptive learning schemes work in a truly behavioral uncertainty problem where interactions matter and agents have the problem of learning about others' behavior.

The problem of forecasting the forecasts of others was originally raised by Townsend (1983). In Townsend's model two representative firms operate respectively in two different but correlated markets. The price in one market is an useful signal about dynamics in the other market. Sargent (1991), Singleton (1987), Kasa (2000) and Pearlman and Sargent (2004) use different approaches to prove that there exists an equilibrium of the "perpetually and symmetrically uninformed" version of the same model of which Townsend's original solution was just an approximation. This equilibrium is the one obtained pooling the information sets held by agents and is independent from higher order beliefs. In practice at equilibrium agents have the same information set and therefore trivially know others' expectations because they know their own. The present work deviates from Townsend (1983) framework in three respects. Firstly, because of the simultaneous cobweb-like timing of expectations, any solutions different from REE only embody the effective influence of first order beliefs, not all higher degree ones (this feature emerges in forward models). Secondly, agents do not have to extract information about exogenous variables from noisy signals of endogenous variable. In the following model, agents are perfectly informed about the exogenous determinants of the economy, and have to extract information about a truly endogenous variable (expectations of others) that directly affects common market fluctuations. Finally we look for an evolutive solution, that is, a real-time dynamic process of convergence of expectations.

The connection of the present work with eductive learning (Guesnerie, 1992; Guesnerie, 2005) is of interest. The problem presented is a typical eductive learning problem

[2] Granato, Guse and Wong (2007) uses a Stackelberg framework in which a group of leading firms has a well specified perceived law of motion and a group of followers just imitate the leaders' forecasts. Again the leaders do not care about heterogeneity, but the followers exploit it. There is no learning about behavioral variables, just one-way imitation.

in that agents perfectly understand the model and its exogenous determinants, but are uncertain about others' actions. What is different here is that we substitute the engine of updating of beliefs. Instead of assuming common knowledge[3], we provide agents with the statistical tools to extract the signal of others' rationality in real time. The exercise in this paper presents methodological innovations in the field of learning and expectations in macroeconomics by showing how to extend the adaptive learning approach to a typical eductive learning problem.

Finally, the nature of the following exercise is very much in the spirit of Adam, Marcet and Nicolini (2008) on internal rationality. There as here, uncertainty about other agents' beliefs drives learning in real time. Nevertheless, the present paper explicitly derives the learning algorithm as the solution to an optimization problem. The underlying strategic structure shaped by the model is also fully spelled out as institutional forecasters' expectations only matter to map the distribution of private sector expectations. Differently, Adam, Marcet and Nicolini (2008) maintain an orthodox general equilibrium perspective in that no agent can influence aggregate quantities and hence the expectations distribution is only mapped by an unknown market function that selects the marginal agent each time.

3.2 Model

3.2.1 The basic framework

A simple self-referential model

Consider an ocean of infinitesimal agents belonging to $A = (0, 1) \subset \mathbb{R}$. They populate a self-referential economy in which output gap at time $t \in \mathbb{N}$, namely $y_t \in \mathbb{R}$, depends jointly on the aggregate expectation on current output gap, labeled as usual $E_{t-1}y_t$, and an exogenous stochastic component. The latter is specified as a $n \times 1$ vector of stationary exogenous variables $\mathbf{x}_t \in \mathbb{R}^n$ weighted by a conform vector of exogenous parameters $\boldsymbol{\alpha}$. This economy is represented by

$$y_t = \boldsymbol{\alpha}'\mathbf{x}_t + \beta E_{t-1}y_t + \eta_t \ , \tag{3.1}$$

[3]The eductive learning approach takes a game-theoretical perspective. Specifically, it analyses under what conditions a REE is a rationalizable solution (Bernheim, 1984; Pearce, 1984) of the game entailed by the self-referential model at hand. In other words, when iterated deletion of strongly dominated strategies, interpreted as recursive updating of higher order beliefs, selects a singleton in strategic space. The dynamics of beliefs evolving in notional time is generated by the key assumption of common knowledge of rationality and of the game.

where $y_t(E_{t-1}y_t) : \mathbb{R} \to \mathbb{R}$ is a linear function going from aggregate expectation to output gap, $\beta \in \mathbb{R}$ measures the impact of aggregate expectation on y_t and η_t is a centred i.i.d. shock normally distributed with variance δ_η. Formally, aggregate expectation is expressed by

$$E_{t-1}y_t = \int_A [y_t]_a^{t-1} \, da \ , \tag{3.2}$$

where $[\cdot]_a^{t-1}$ is a function belonging to a family of generic belief operators $\mathcal{F}$, whose image is agent a's belief at time $t-1$ of $(\cdot)$, with $a \in A$. The unique rational expectation for the model at hand is $\overline{y}_t = \boldsymbol{\alpha}'\mathbf{x}_t \left(1-\beta\right)^{-1}$ yielding the unique rational expectation equilibrium $y_t^* = \overline{y}_t + \eta_t$.

Law (3.1) can be recovered as the reduced form of a fully specified model where an profile of expectations is mapped in outcomes by actions[4]. Nevertheless, rationality (in the Muthian sense) of the aggregate expectation does not imply rationality of agents. The Microfoundation of this notion should rely on consistent specification of agents' individual incentives. Let us, therefore, assume all agents have to minimize the following loss function

$$\Lambda_a = \mathbf{E}\left(y_t - [y_t]_a^{t-1}\right)^2 \ , \tag{3.3}$$

where $[y_t]_a^{t-1}$ is the belief of agent a at time $t-1$ of y_t, and $\mathbf{E}[\cdot]$ is the mathematical expectation operator. The introduction of a loss function provides agents with explicit motives to consistently forecast on y_t. This hypothesis is generally not formally spelled out, though it is always implicitly assumed. It is clear from simple inspection of the problem that rationality of the aggregate expectation can only be sustained by all agents holding rational expectation. According to (3.3), deviations from the rational expectation would entail a systematic loss. In other words, individual rationality and the self-referential nature of the economy imply a coordination game among agents whose equilibrium is the REE if and only if everyone expects it.

This concludes the description of a very simple model often employed as benchmark in economic dynamics literature. For instance, it is consistent with the model proposed by Muth (1961) in his seminal paper on rational expectations or Lucas's (1973) framework. This will guarantee transparent comparison of the proposed approach with the existing literature. However, we do not commit to any specific model in order to keep the dynamic analysis free from constraints linked to the particular economic theory embodied in the reduced form. For the scope of the present work, this minimal economy is enough to make the point. See Gaballo (2009) for an elaboration on the findings of this paper in a

[4]For example, in Gaballo (2009a) this reduced form is obtained at the limit of a Cournot game in which suppliers choose quantity a period before, according to their expectation on current output. Other examples are provided by Branch and Evans (2007) and Woodford (2003).

fully specified Lucas-type monetary model.

Polarized expectations

A standard perspective in macroeconomics is that all agents are equal and have equally negligible effect on aggregate variables. More realistically, we can imagine that individual expectations are not independent. The reader will agree that most part of agents do not actually have a theory on how the economy works. Very few agents are indeed able to produce statistically correct expectations by exploiting available information. A brief list of institutional forecasters would include market leaders, monetary and fiscal authorities, rating agencies and financial institutes. The latter are the only agents holding procedural rationality and they act as focal points for the private sector.

Let us formally define this form of structural heterogeneity, assuming institutional forecasters beliefs are formed according to

$$E_{t-1}^{i} y_t \equiv \mathbf{E}[y_t|\Omega_{t-1}^{i}], \quad \forall i \in A^*, \tag{3.4}$$

with $A^* \subset A$ being the set of institutional forecasters in the economy where $E_{t-1}^{i} \in \mathcal{F}$. Institutional forecasters maintain mathematical expectation of the generic process x_t conditioned to available information up to time $t-1$. Assumption (3.4) is a formal specification of procedural rationality. It is natural to think institutional forecasters are very few because information processing has strong scale economy effects. On the other hand, the private sector has the following expectation function

$$[y_t]_{t-1}^{z} = E_{t-1}^{i_z} y_t, \quad \forall z \in A/A^*, \tag{3.5}$$

where $i_z \in A^*$ is the institutional forecaster on which agent z relies, and $[\cdot]_{t-1}^{z} \in \mathcal{F}$ is merely an imitation correspondence. In this sense, the information transmission channel from institutional forecasters to the private sector is assumed neutral. If this working hypothesis is reasonable, agents' expectations are polarized around few institutional forecasters' forecasts. Given (3.5), the aggregate expectation in the economy is

$$E_{t-1} y_t = \sum_{i \in A^*} \lambda_i E_{t-1}^{i} y_t \ , \quad \sum_{i \in A^*} \lambda_i = 1, \tag{3.6}$$

where $\lambda_i \in (0,1)$ is the fraction of the private sector relying on agent i's expectation. Agent i's reliability measures the average impact of agent i's expectation on the aggregate expectation.[5] λ_i are assumed exogenous for the sake of simplicity. Introduction of an

[5] One uncomfortable with the mere imitation implied by (3.5) can assume more sophisticated agents weighting all institutional forecaster's expectations. From this perspective λ_i would be interpreted as the

endogenous credibility mechanism would be an interesting but cumbersome extension with respect the truly aim of this paper.

Equation (3.6) invalidates the negligibility of agents' individual impact in the economy, as assumed in the general equilibrium perspective. In particular, the impact of each institutional forecaster in the economy is proportional to its " audience ". Nevertheless, occurrence of the REE depends uniquely on the coordination game played by institutional forecasters (entailed by (3.1),(3.3) and (3.6)). Moreover, rational expectation is the best expectation if and only if all others do the same.

3.2.2 Best expectation functions

Since the task is to study the basic dynamics generated by agents trying to solve behavioral uncertainty in real time, we focus on this very problem to isolate the minimal requirements triggering the basic mechanism of interest. We keep aside, as first approximation, issues concerning learning about the exogenous determinants of the economy that had already been extensively studied in previous adaptive learning literature. So, we assume institutional forecasters to maintain perfect knowledge of the underlying structure of the economy $y_t(\cdot)$ and of all relevant exogenous variables as they emerge in real time. This is a standard hypothesis in eductive learning literature, that shares the object of investigation with this work. We also assume that the set A^* and functional form of (3.6) are known by agents. That is, institutional forecasters' expectations are public information and their role as focal point is acknowledged (but not necessarily commonly known). What institutional forecasters do not know is only behavioral data, namely, how agents form their expectations and/or the shape of agents' loss functions (3.3). In this sense agents are epistemically isolated and the problem is as a pure behavioral uncertainty problem.

Since institutional forecasters are perfectly informed about the structural form of the model, but not about other agents' beliefs, they have the following expectation function:

$$E^j_{t-1} y_t = (1-\beta)\,\overline{y}_t + \beta \sum_{i \in A^*} [\lambda_i]^j_{t-1}\, E^j_{t-1} E^i_{t-1} y_t, \tag{3.7}$$

for each $j \in A^*$. The generic beliefs operator $[\cdot]^j_{t-1}$ for λ_i is need to emphasize that no optimal property is required for beliefs on λ_i in the following arguments. By (3.7), institutional forecasters are aware that the actual dynamics of the economy will be stuck around the REE if and only if all institutional forecasters have rational expectations.

average consideration of agent i in expectations formation of the private sector. This would not change the substance and mathematics of the following arguments.

Remark 15 *Notice that unlike the standard adaptive learning approach, it is explicitly assumed that institutional forecasters acknowledge the self-referential nature of the economy. In this sense they are not rationally bounded as usually assumed in adaptive learning literature.*

Institutional forecasters know the fundamental exogenous process $\overline{y}_t$, but, like any agent in the economy, they are not interested in the fundamental *in se*. Their only concern is to have consistent forecasts of the actual course of the economy. Thus institutional forecasters have to take their own power to displace the actual course far away from the REE into account. They form *best expectations* according to

$$E^i_{t-1}y_t = \frac{1-\beta}{1-\beta\,[\lambda_i]^i_{t-1}}\overline{y}_t + \frac{\beta}{1-\beta\,[\lambda_i]^i_{t-1}}\sum_{j\in A^*/\{i\}}[\lambda_j]^i_{t-1}\,E^i_{t-1}E^j_{t-1}y_t, \tag{3.8}$$

for each $i \in A^*$. Rule (3.8) is a best response function conditioned to a given (possibly wrong) others' expectations profile. Notice that (3.8) depends strictly on the fact that agents have procedural rationality and know exogenous determinants of the economy, but it does not depend at all on any common knowledge property. In other words, this framework properly embodies the meaning of Bayesian rationality. Given the above hypothesis, what really matters for solving agents' maximization problem is knowledge of others' simultaneous expectations. The next subsection will shed light on this point.

Perfect knowledge of others' simultaneous expectations

For the sake of clarity and to maintain a full analytical analysis, let us assume from here onward there are only two professional forecasters in the economy, namely $A^* = \{1, 2\}$. Rearranging the agents' best expectations functions according to (3.8):

$$E^1_{t-1}\left(y_t - \overline{y}_t\right) = [\theta_1]^1_{t-1}\left(E^1_{t-1}E^2_{t-1}y_t - \overline{y}_t\right), \tag{3.9a}$$

$$E^2_{t-1}\left(y_t - \overline{y}_t\right) = [\theta_2]^2_{t-1}\left(E^2_{t-1}E^1_{t-1}y_t - \overline{y}_t\right), \tag{3.9b}$$

where $[\theta_i]^i_{t-1} \equiv \beta\left(1-[\lambda_i]^i_{t-1}\right)\left(1-\beta[\lambda_i]^i_{t-1}\right)^{-1}$. Behavioral consistency holds for each $i \in A^*$ when:

$$E^i_{t-1}y_t \in \Omega^j_{t-1} \quad \text{with} \quad i \neq j\ , \tag{3.10}$$

which means that agent i knows agent j's simultaneous expectation on the current output gap at time $t-1$. Note that $[y_t]^i_{t-1}$ is a $t-1$ datum and that $E^i_{t-1}x_t \in \Omega^i_{t-1}$ is always trivially true. To see that (3.10) is really what agents need to solve their problem optimally let us assume absence of behavioral uncertainty, namely $\Omega^i_{t-1} = \Omega^j_{t-1}$. The iterated expectations law implies that

$$E^j_{t-1}E^i_{t-1}y_t = \mathbf{E}[\mathbf{E}[y_t|\Omega^i_{t-1}]|\Omega^j_{t-1}] = E^i_{t-1}y_t. \tag{3.11}$$

Thus, solving the linear system (3.9) we have

$$E^1_{t-1}y_t = E^2_{t-1}y_t = \overline{y}_t,$$

irrespective of $[\theta_1]^{t-1}_1$ and $[\theta_2]^{t-1}_2$. In other words, knowledge of others' simultaneous expectations plus individual rationality implies the aggregate expectation is the rational one whether or not agents' beliefs on their impact are correct. knowledge of others' simultaneous expectations plus rationality implies the aggregate expectation is the rational one. Otherwise, the extent of displacements of actual dynamics from REE depend on $[\theta_i]^i_{t-1}$. The two-agents case can be easily extended to a numerable set of agents without changing the conclusions. This is not surprising since, if agents are all rational *and* they hold perfect knowledge of the model and others' first order beliefs, they maintain rational expectations.[6]

Behavioral uncertainty

Condition (3.10) is a very strong requirement. Knowledge of simultaneous expectations (actions) is the hypothesis implicit in a "Robinson Crusoe" economy populated by a single representative agent, in which coordination failures are avoided by internalizing externalities. To introduce behavioral uncertainty in the model is to specify an epistemic system in which (3.10) is violated. Breaking (3.10) generates heterogeneity of agents in the economic system. This kind of heterogeneity does not require different forecasting functions (structural heterogeneity) or different priors, but arises from the sole fact that agents are separate individuals and therefore they cannot have instantaneous knowledge of simultaneous actions. We will refer to this kind of heterogeneity as *intrinsic heterogeneity.*

Instead of considering agents' instantaneous perfect knowledge of others' simultaneous expectations, as stated in (3.11), let us assume more realistically that they commit observational errors in perceiving others' simultaneous expectations. For instance, agents can apply margins of truly subjective interpretation to public information.[7] The simultaneous nature of agents' expectations generally means that they fail to have instantaneous perfect information of what the other is expecting at the same time. We chose to model

[6] See Aumann and Brandenburger (1995) for a formal statement of this proposition in a proper game-theoretic framework.

[7] See Bullard, Evans, Honkapohja (2008) for a theoretical study of the effects of such hypothesis in a learning model of asset pricing.

behavioral uncertainty, assuming that agents commit a stochastic observational error in perceiving others' simultaneous expectations. We can formally define intrinsic heterogeneity as

$$\left(E_{t-1}^{i}E_{t-1}^{j}y_t - E_{t-1}^{j}y_t\right) \equiv v_{i,t-1} \sim \Upsilon_i\left(\mu_i, \delta_i\right) \tag{3.12}$$

where $\Upsilon_i\left(\mu_i, \delta_i\right)$ is a generic distribution with finite mean μ_i and variance δ_i. We also assume the exclusive subjective nature of observational errors, so we assume them to be independent from exogenous determinants of the model $\mathbf{x}_t$ and η_t. They can be interpreted of observational errors, or simply as first order beliefs, or as subjective judgemental adjustments. We do not subscribe to any positive theory about why agents make measurement errors or why they have biased information about others' expectations. We are not concerned with how agents can correct observational errors (if any). Assumption (3.12) just encompasses a general failure of (3.10), that is, whenever behavioural uncertainty reveals, a specification of intrinsic heterogeneity (3.12) describing the actual setting there should exists.[8] This framework is extremely general and can account for very different specifications of the basic problem. In any case, (3.12) implies behavioral uncertainty among agents is stationary in some specific way linked to the choice of $\Upsilon_i\left(\mu_i, \delta_i\right)$.

Summing up, we can formally state that the information set held by institutional forecaster i at time t

$$\Omega_t^i \equiv (y_t(\cdot), \{\mathbf{x}_\tau\}_{\tau=0}^{t}, \{y_\tau\}_{\tau=0}^{t}, \{E_\tau^j y_{\tau+1} + v_{i,\tau}\}_{\tau=1}^{t}), \tag{3.13}$$

is composed by: i) the exact perception of the model underlying the economy (coefficients included), $y_t(\cdot)$; ii) the history of exogenous variables $\mathbf{x}$ (this is enough to compute the rational expectation $\overline{y}_t$), $\{\mathbf{x}_\tau\}_{\tau=0}^{t-1}$; iii) the history of actual output gaps, $\{y_\tau\}_{\tau=0}^{t-1}$; and vi) the history of other's perceived expectations up to time t, $\{E_\tau^j y_{\tau+1} + v_{i,\tau}\}_{\tau=1}^{t}$. Noisy observations defined by (3.12) are the best proxy for agents' actual behavior.[9]

In such a case, institutional forecasters' best expectations are given by

$$E_{t-1}^{1}\left(y_t - \overline{y}_t\right) = \widehat{b}\left(E_{t-1}^{2}y_t + v_{1,t-1} - \overline{y}_t\right), \tag{3.14a}$$

$$E_{t-1}^{2}\left(y_t - \overline{y}_t\right) = \widehat{c}\left(E_{t-1}^{1}y_t + v_{2,t-1} - \overline{y}_t\right), \tag{3.14b}$$

[8] One can also maintain a long-run interpretation regarding observational errors as having more an ontological root rather than an epistemic one. In other words, heterogeneity of agents is defined through (3.12) and observational errors exist as long as agents are defined as different decision units.

[9] Keep in mind that agent i cannot disentangle $E_{t-1}^{j}y_t$ from his perception of agent j's simultaneous expectation $E_{t-1}^{j}y_t + v_i$ since $E_{t-1}^{j}y_t \notin \Omega_{t-1}^{i}$.

where

$$\widehat{b} \equiv \frac{\mathbf{E}[(y_t - \overline{y}_t)\left(E^2_{t-1}y_t - \overline{y}_t + v_{1,t-1}\right)]}{\mathbf{E}\left(E^2_{t-1}y_t - \overline{y}_t + v_{1,t-1}\right)^2}, \tag{3.15a}$$

$$\widehat{c} \equiv \frac{\mathbf{E}[(y_t - \overline{y}_t)\left(E^1_{t-1}y_t - \overline{y}_t + v_{2,t-1}\right)]}{\mathbf{E}\left(E^1_{t-1}y_t - \overline{y}_t + v_{2,t-1}\right)^2}, \tag{3.15b}$$

are respectively solutions to the first order conditions of $\min_b \Lambda_1$ and $\min_c \Lambda_2$. In words, $(\widehat{b}, \widehat{c})$ are coefficients for which agents' expectations are optimal linear projection given available information Ω^i_{t-1}. Noisy information about other's expectations are taken in to account, that is $(\widehat{b}, \widehat{c}) \neq (0,0)$, as long as they provide in some extent information on displacements of actual output gap form the rational expectation.

The reader can easily prove that $\widehat{c} = 0$ implies $\widehat{b} = 0$ and viceversa, given in this case $\mathbf{E}[(E^i_{t-1}y_t - \overline{y}_t)v_{i,t-1}] = 0$. For $(\widehat{b}, \widehat{c}) = (0,0)$ the rational expectation is the best expectation, and hence the aggregate expectation is rational as well. This means that agents do not modify their expectations on noisy observations of the other's simultaneous expectations, as it is redundant information. Intuitively, if this was not the case, weighting noisy observations would improve forecast accuracy. In fact, if at least one institutional forecaster has non rational expectations, agents observe actual deviations from the rational expectation that are not only due to the exogenous unpredictable noise of the model η_t. Summing up, $(\widehat{b}, \widehat{c})$ measures the optimal interdependency of institutional forecasters' expectations. However the degree of optimal interdependency is intrinsically indeterminate due to the referential nature of the expectations formation process. The existence of values $(\widehat{b}, \widehat{c}) \neq (0,0)$ satisfying (3.15) cannot be excluded trivially. The existence of such solutions is assessed in next section.

3.3 Equilibria and Learning Analysis

3.3.1 Equilibria

Computation of equilibria as fixed points of the T-map

Before computing equilibria, let us do some simple manipulation. Initially, consider agents have possibly non optimal expectations, formed according to

$$E^1_{t-1}(y_t - \overline{y}) = b\left(E^2_{t-1}y_t + v_{1,t-1} - \overline{y}_t\right), \tag{3.16a}$$

$$E^2_{t-1}(y_t - \overline{y}) = c\left(E^1_{t-1}y_t + v_{2,t-1} - \overline{y}_t\right), \tag{3.16b}$$

where possibly $b \neq \widehat{b}$ and $c \neq \widehat{c}$. From (3.16) it is easy to check that best expectation functions can be expressed as linear functions of the fundamental output gap and observational errors. Formally we have

$$E^1_{t-1}(y_t - \overline{y}_t) = \frac{bc}{1-bc}v_{2,t-1} + \frac{b}{1-bc}v_{1,t-1} , \tag{3.17a}$$

$$E^2_{t-1}(y_t - \overline{y}_t) = \frac{bc}{1-bc}v_{1,t-1} + \frac{c}{1-bc}v_{2,t-1} , \tag{3.17b}$$

provided $bc \neq 1$. Processes (3.17) cannot be inferred by agents since they cannot distinguish observational errors. According to (3.1), (3.6) and (3.17) the law of motion of displacements of the actual output gap from the fundamental one is

$$y_t - \overline{y}_t = \beta\left(\frac{b(\lambda + (1-\lambda)c)}{1-bc}v_{1,t-1} + \frac{c(\lambda b + (1-\lambda))}{1-bc}v_{2,t-1}\right) + \eta_t. \tag{3.18}$$

For $(b,c) = (0,0)$, (3.17) reduces to the rational expectation and (3.18) reduces to the REE. Note, that processes different from REE have the feature of including behavioral variables, namely observational errors, among those affecting the course of actual output gap. Specifically, as $(b,c) \neq (0,0)$ deviations from the fundamental output gap do not depend solely on exogenous noise η_t, but also on $v_{1,t-1}$ and $v_{2,t-1}$. Equilibria of the model are described by (3.18) and (3.17) with $(b,c) = \left(\widehat{b},\widehat{c}\right)$. For such values, agents' loss functions (3.3) are stuck in a local minimum. Let us introduce the following definition.

Definition 16 *Rational expectation equilibrium (REE) is a sequence of* $\{y_t\}$ *according to (3.18) such that* $(b,c) = \left(\widehat{b},\widehat{c}\right) = (0,0)$. *Behavioral sunspots equilibrium (BSE) is a sequence of* $\{y_t\}$ *according to (3.18) such that* $(b,c) = \left(\widehat{b},\widehat{c}\right) \neq (0,0)$.

The word sunspot refers to the presence of an extra stochastic component in the agents' expectation function, namely observational errors. Like any sunspot solution, BSEs exhibit extra volatility triggered by exogenous variables of no economic relevance, in this case, observational errors. The extra variance is

$$\beta^2\left(\frac{\widehat{b}^2(\lambda+(1-\lambda)\widehat{c})^2}{\left(1-\widehat{b}\widehat{c}\right)^2}\delta_1+\frac{\widehat{c}^2\left(\lambda\widehat{b}+(1-\lambda)\right)^2}{\left(1-\widehat{b}\widehat{c}\right)^2}\delta_2+2\frac{\widehat{b}\widehat{c}\left(\lambda\widehat{b}+(1-\lambda)\right)(\lambda+(1-\lambda)\widehat{c})}{\left(1-\widehat{b}\widehat{c}\right)^2}\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right) \tag{3.19}$$

and any drift is

$$\beta\left(\frac{\widehat{b}(\lambda+(1-\lambda)\widehat{c})}{1-\widehat{b}\widehat{c}}\mu_1 + \frac{\widehat{c}(\lambda\widehat{b}+(1-\lambda))}{1-\widehat{b}\widehat{c}}\mu_2\right), \tag{3.20}$$

in case observational errors embody a deterministic component (one can also think it as deterministic fluctuation rather than a constant). In any case, no assumption in this model implies professional forecasters have a drift on others' simultaneous expectations. The framework is designed to be as general as possible.

In order to compute equilibria coefficients $\widehat{b}$ and $\widehat{c}$, let us spell out first order conditions of the minimization problems $\min_b \Lambda_1$ and $\min_c \Lambda_2$. They are

$$\mathbf{E}[\phi_{t-1}^1 \left(y_t - \overline{y}_t - T_b\left(b,c\right)\ \phi_{t-1}^1\right)] = 0 \tag{3.21a}$$

$$\mathbf{E}[\phi_{t-1}^2 \left(y_t - \overline{y}_t - T_c\left(b,c\right)\ \phi_{t-1}^2\right)] = 0 \tag{3.21b}$$

with $\phi_{t-1}^i = E_{t-1}^j y_t - \overline{y} + v_{i,t-1}$, where $T_b(b,c)$ and $T_c(b,c)$ are best linear projection coefficients for possibly non optimal coefficients b and c. Therefore equilibria coefficients are obtained as fixed points of the T-map for $T_b\left(\widehat{b},\widehat{c}\right) = \widehat{b}$ and $T_c\left(\widehat{b},\widehat{c}\right) = \widehat{c}$.

Remark 17 *Conditions (3.21) are those used in Marcet and Sargent (1989a, 1989b) and Sargent (1991) to compute partially informed rational expectation equilibria. Note that orthogonality conditions yielding the T-map arise if agents minimize the loss function (3.3).*

As the intrinsic indeterminacy of expectations is solved expressing them as linear combinations of the rational expectation and observational shocks, we are now able to compute the T-map.

Proposition 18 *The T-map takes the form*

$$T_b = \beta\left(\frac{b\left(\lambda+\left(1-\lambda\right)c\right)\left(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+c\left(\lambda b+\left(1-\lambda\right)\right)\left(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_1+\varepsilon_2 c^2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right),$$

$$T_c = \beta\left(\frac{c\left(\lambda b+\left(1-\lambda\right)\right)\left(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+b\left(\lambda+\left(1-\lambda\right)c\right)\left(b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_2+\varepsilon_1 b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right),$$

where $\varepsilon_i = \mathbf{E}[v_{i,t}^2]$ *and* $\rho_v \equiv \mathbf{E}[v_{1,t}v_{2,t}]/\sqrt{\varepsilon_1\varepsilon_2}$.

Proof. See appendix A1. ■

Notice that for $\mu_i = 0$, ρ_v is exactly the correlation ratio between the contemporaneous observational errors of institutional forecasters. The T-map depends on the ratio $\varepsilon_1/\varepsilon_2$ between second moments of the distributions of observational errors and not on their extent. Moreover, if $\varepsilon_1 = \varepsilon_2$ error variances simply disappear from the T-map equation. Fixed points of the T-map are such that professional forecasters do not commit systematic errors given available information. The fixed point are solution to the system:

$$\widehat{b} = \frac{\beta\widehat{c}(1-\lambda)(\widehat{c}\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{((1-\beta\lambda)\varepsilon_2-\beta(1-\lambda)\sqrt{\varepsilon_1\varepsilon_2}\rho_v)\widehat{c}^2+(2(1-\beta\lambda)\sqrt{\varepsilon_1\varepsilon_2}\rho_v-\beta(1-\lambda)\varepsilon_1)\widehat{c}+(1-\beta\lambda)\varepsilon_1} \quad (3.22a)$$

$$\widehat{c} = \frac{\beta\widehat{b}\lambda(\widehat{b}\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{((1-\beta(1-\lambda))\varepsilon_1-\beta\lambda\sqrt{\varepsilon_1\varepsilon_2}\rho_v)b^2+(2(1-\beta(1-\lambda))\sqrt{\varepsilon_1\varepsilon_2}\rho_v-\beta\lambda\varepsilon_2)\widehat{b}+(1-\beta(1-\lambda))\varepsilon_2} \quad (3.22b)$$

assuming $\widehat{bc} \neq 1$. It is easy to prove the following

Proposition 19 *The unique REE for* $(\widehat{b}, \widehat{c}) = (0, 0)$ *is always an equilibrium of the system.*

The proposition above can simply be proved by substitution in (3.22).

Remark 20 *Note that the proposition contemplates all possible values of the coefficients of the system. In particular it is true even if agents do not have consistent expectations about other's expectations, that is, if they still commit systematic errors in their perception of others' simultaneous expectations (the case* $\mu_i \neq 0$*).*

Existence of Behavioral Sunspot Equilibria

Now let us consider existence of BSEs. Unfortunately, it is not possible to provide a closed-form solution for all values. To assess the existence of BSEs analytically we focus on the symmetric agents case $\lambda = 1/2$, $\varepsilon_1 = \varepsilon_2$ with independent observational errors ($\rho_v = 0$). Numerical investigation of all other possible BSEs solutions will be the object of a distinct study.

In the symmetric case, the domain of possible solutions is restricted to the space $|\widehat{b}| = |\widehat{c}|$. This is appraised from inspection of equations (3.22). The system (3.22) is now expressed by

$$\widehat{c}\left((1-\beta/2)\,\widehat{c}^2 - \beta\widehat{c} + (1-\beta/2)\right) = 0. \quad (3.23)$$

Its solutions other than $(0, 0)$ are[10] high BSE (b_+, c_+) and a low BSE (b_-, c_-) where

$$\widehat{c}_+ = \widehat{b}_+ = \frac{\beta + 2\sqrt{(\beta-1)}}{2-\beta}, \quad (3.24a)$$

$$c_- = \widehat{b}_- = \frac{\beta - 2\sqrt{(\beta-1)}}{2-\beta}, \quad (3.24b)$$

that is, at least one non fundamental solution arises for $\beta \geq 1$. The reader can easily prove[11] $\mathrm{d}\widehat{c}_+ \backslash \mathrm{d}\beta > 0$ and $\mathrm{d}\widehat{c}_+ \backslash \mathrm{d}\beta < 0$. Since $\beta = 1$ implies $\widehat{c}_- = \widehat{c}_+ = 1$, and since

[10]Notice that solutions $\left(\widehat{b}_+, \widehat{c}_-\right)$ and $\left(\widehat{b}_-, \widehat{c}_+\right)$ are excluded a-priori by the requirement $\widehat{bc} \neq 1$.

[11]Exploit the fact that $\beta - 2\sqrt{(\beta-1)} > 0$ is always true.

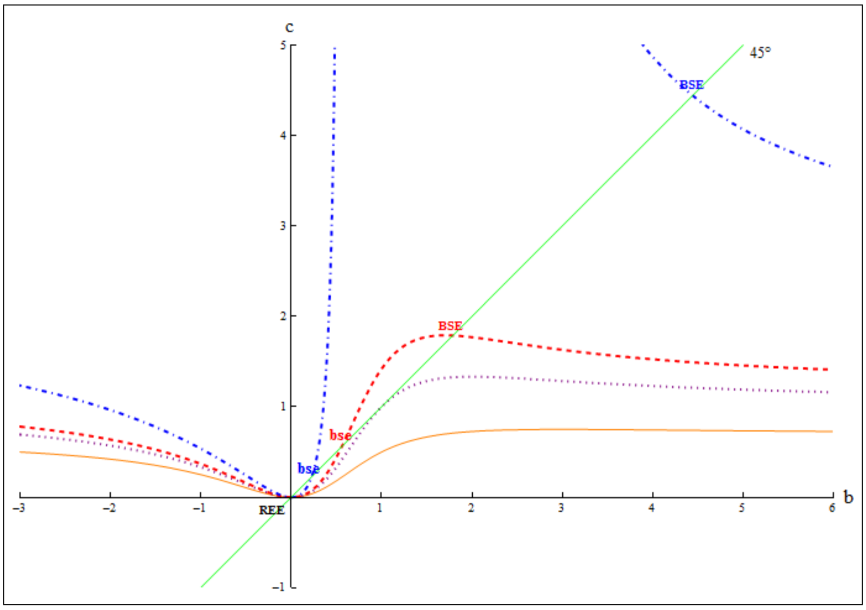

Figure 3.1: T-map for the case of two symmetrical agents ($\lambda = 0.5$ and $\varepsilon_1 = \varepsilon_2$) and no correlation between observational errors $\rho_v = 0$. Curves are obtained for different values of β: 0.8 normal, 1 dotted, 1.08 dashed, 1.4 dot-dashed. "BSE" denotes the high behavioral sunspot equilibrium and "bse" the low one.

$\lim_{\beta \to 2_-} \widehat{c}_+ = +\infty$, and $\lim_{\beta \to 2_-} \widehat{c}_- = 0$, the low (high) BSE is monotonically decreasing (increasing) in β and collapses to REE (goes to infinity) as β approaches 2.

Some examples are represented in figure 1. The curve plots the right hand side of equation (3.22a) with $\lambda = 0.5$, $\varepsilon_1 = \varepsilon_2$ and $\rho_v = 0$ for different values of β, namely 0.8, 1, 1.08, 1.4, in the space (b, c). Intersections of this curve and the bisector are fixed points of the T-map in case of two symmetrical professional forecasters having audiences of the same size.

Proposition 21 *Solutions different from the REE, specifically two BSEs corresponding to (3.24), exist for the symmetric case* ($\lambda_1 = 0.5$ *and* $\varepsilon_1 = \varepsilon_2$) *with* $\rho_v = 0$ *provided* $\beta \in [1, 2)$.

According to a continuity argument, we can extend the proposition to a non trivial open region of parameter space.

Remark 22 *The key feature making REE and BSEs coexist is the non-linearity of the T-map. Notice that BSEs are kind of limited-informed rational expectation (Sargent 1991), because agents cannot observe all the stochastic components of the actual law of motion separately. However, unlike the standard setting proposed there, here the subjective information set* $\Omega_{i,t}$ *results in a coarser set, rather than a subset, of all the relevant exogenous* *variables. This occurs because observational errors are correlated by unavoidable non linear constraints resulting from the system (3.16). This technical feature underlies the non-linearity of the T-map and the emergence of BSEs.*

At this point it is worthwhile discussing the economic insights underlying the existence of BSEs with reference to classical sunspot equilibria. BSEs and classical sunspot equilibria are both determined by additional variables uncorrelated with exogenous variables of the model. They differ in the kind of beliefs required to self-sustain them. In classical sunspot equilibria agents are required to hold common knowledge that a certain observable extra variable affects the equilibrium. On the other hand, BSEs require agents to believe that displacements of the actual output gap from the fundamental one depend on others' deviations from REE prescriptions. It is not required that they believe that others also believe. Moreover BSEs are not rational expectation equilibria since mathematical expectation of the actual law of motion at time $t-1$ does not match any individual expectation, though they differ by a stochastic centred factor in case $\mu_i = 0$.

BSEs take shape as a classical coordination failure. After a certain threshold, a best action for agent 1 is to condition his own expectations to noisy observations of agent 2's expectations since agent 2 is doing the same. Specifically, an agent conditioning his expectations on noisy signals of the other agent's simultaneous expectation results appears as "irrational exuberant"; in turn, an agent assessing irrational exuberance of the other agent has incentive to modify his expectation according to the noisy observation of the other agent's simultaneous expectations. When a BSE is achieved, variance of observational errors transmits persistently to the course of actual output gap through aggregate expectation. This mechanism provides new insights on how behavioral uncertainty can trigger excess volatility phenomena. The exercise at hand proves the issue is of interest *even if* agents know everything about the exogenous determinants of the economy.

3.3.2 Learnability

This section explores learnability of REE and the possibility of adaptive learners being stuck in a BSE. Institutional forecasters can exploit available information to extract the signals of others' rationality by running recursive least square regressions on available data. Note that OLS regression is the solution of the minimization problem of the forecast error variance as stated in (3.3). Let us suppose institutional forecasters estimate $\widehat{b}$ and $\widehat{c}$ in real time with a standard ordinary least square regression, having the following recursive form

$$
\begin{aligned}
b_t &= b_{t-1} + t^{-1} R_{1,t}^{-1} \left(E_{t-1}^2 \left(y_t - \overline{y}_t \right) + v_{1,t-1} \right) \left(y_t - \overline{y}_t - E_{t-1}^1 \left(y_t - \overline{y}_t \right) \right), && \text{(3.25a)} \\
R_{1,t} &= R_{1,t-1} + t^{-1} \left(\left(E_{t-1}^2 \left(y_t - \overline{y}_t \right) + v_{1,t-1} \right)^2 - R_{1,t-1} \right), && \text{(3.25b)} \\
c_t &= c_{t-1} + t^{-1} R_{2,t}^{-1} \left(E_{t-1}^1 \left(y_t - \overline{y}_t \right) + v_{2,t-1} \right) \left(y_t - \overline{y}_t - E_{t-1}^2 \left(y_t - \overline{y}_t \right) \right), && \text{(3.25c)} \\
R_{2,t} &= R_{2,t-1} + t^{-1} \left(\left(E_{t-1}^1 \left(y_t - \overline{y}_t \right) + v_{2,t-1} \right)^2 - R_{2,t-1} \right), && \text{(3.25d)}
\end{aligned}
$$

so that, they form expectations according to the rule

$$
\begin{aligned}
E_{t-1}^1 \left(y_t - \overline{y}_t \right) &= b_{t-1} \left(E_{t-1}^2 \left(y_t - \overline{y}_t \right) + v_{1,t-1} \right), && \text{(3.26a)} \\
E_{t-1}^2 \left(y_t - \overline{y}_t \right) &= c_{t-1} \left(E_{t-1}^1 \left(y_t - \overline{y}_t \right) + v_{2,t-1} \right). && \text{(3.26b)}
\end{aligned}
$$

The concept of learnability refers to the stable or unstable nature of the learning dynamics under a recursive least square algorithm around the equilibria computed above.

Definition 23 *An equilibrium* $(\widehat{b}, \widehat{c})$ *is locally learnable under a recursive least square (RLS) algorithm if and only if there exists some neighborhood* $\Im(\widehat{b}, \widehat{c})$ *of* $(\widehat{b}, \widehat{c})$ *such that for each initial condition* $(b_0, c_0) \in \Im(\widehat{b}, \widehat{c})$ *the estimates converge almost surely to the equilibrium, that is* $\lim_{t\to\infty} b_{t-1} \overset{a.s.}{=} \widehat{b}$ *and* $\lim_{t\to\infty} c_{t-1} \overset{a.s.}{=} \widehat{c}$*.*

To check learnability it is necessary to investigate the Jacobian of the T-map. If the matrix of all partial derivatives of the T-map in the equilibrium has all eigenvalues in the unit root, we can say that the equilibrium is stable under learning (Marcet and Sargent 1989, Evans Honkapohja 2001). The Jacobian for the T-map takes the form

$$
JT(b,c) = \begin{pmatrix} \frac{dT_b(b,c)}{db} & \frac{dT_b(b,c)}{dc} \\ \frac{dT_c(b,c)}{db} & \frac{dT_c(b,c)}{dc} \end{pmatrix}
$$

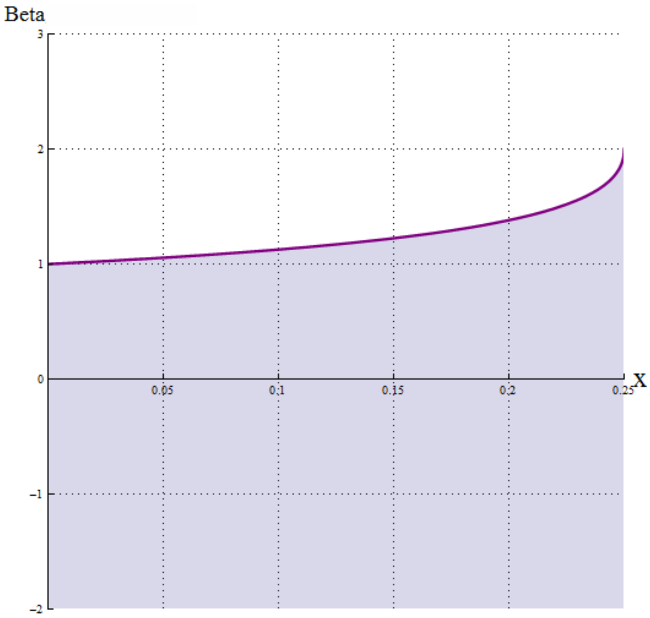

Figure 3.2: REE Learnability. The shadow area denotes the learnability region in parameter space $(\beta^*, \rho_v, \lambda)$ for any pair $(\varepsilon_1, \varepsilon_2)$ where $x \equiv \lambda(1-\lambda)(1-\rho_v^2)$.

where

$$\begin{aligned}
\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}b} &= \beta\frac{(\lambda+(1-\lambda)c)(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v)+c\lambda(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{\varepsilon_1+\varepsilon_2c^2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v}, \\
\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}c} &= \beta\frac{b(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(1-\lambda)+b(\sqrt{\varepsilon_1\varepsilon_2}\rho_v c(1-\lambda)+\lambda)+2c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(1-\lambda+\lambda b)}{\varepsilon_1+\varepsilon_2c^2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v} - \\
&\quad -\beta\frac{(2c\varepsilon_2+2\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(b(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(c(1-\lambda)+\lambda)+c(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v(1-\lambda+\lambda b))}{(\varepsilon_1+\varepsilon_2c^2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v)^2}, \\
\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}b} &= \beta\frac{c(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v)\lambda+c(\sqrt{\varepsilon_1\varepsilon_2}\rho_v\,(b\lambda+1-\lambda)+2b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(\lambda+(1-\lambda)\,c)}{\varepsilon_2+\varepsilon_1b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v} - \\
&\quad -\beta\frac{(2b\varepsilon_1+2\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(c(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v)(b\lambda+1-\lambda)+b(b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v(\lambda+(1-\lambda)\,c))}{(\varepsilon_2+\varepsilon_1b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v)^2}, \\
\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}c} &= \beta\frac{(1-\lambda+\lambda b)(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v)+b(1-\lambda)(b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v)}{\varepsilon_2+\varepsilon_1b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v}.
\end{aligned}$$

The following is true.

Proposition 24 *The REE solution* $(\widehat{b}, \widehat{c}) = (0,0)$ *is learnable whenever*

$$\beta \leq \frac{1-\sqrt{1-4\lambda(1-\lambda)\,(1-\rho_v^2)}}{2\lambda(1-\lambda)\,(1-\rho_v^2)}. \tag{3.27}$$

Proof. See appendix A2. ■

The shadow area under the bold curve in figure 2 denotes the learnability region for the unique REE, whereas in the white region no convergence occurs. Moreover for $\rho_v = 0$ (that implies $\mu_i = 0$), condition (3.27) collapses to

$$\max[\beta\lambda_1, \beta(1-\lambda_1)] < 1. \tag{3.28}$$

These results give us conditions under which expectations consistency can be met for the model at hand *as if* (3.10) held. Note again that (3.27) does not depend on second moment absolute values of observational error distributions but only on the ratio between them. Recall that such observational errors were defined in a very general way by (3.12). In particular REE learnability holds in a region greater than $\beta < 1$ even if observational errors are correlated or have systematic bias. Nevertheless, learnability of REE does not hold in the whole parameter space.

Now let's analyse learneability of BSEs computed for the symmetric case.

Proposition 25 *In the symmetric case* ($\lambda_1 = 0.5$ *and* $\varepsilon_1 = \varepsilon_2$) *with* $\rho_v = 0$*, whenever BSEs exist, the high one is always learnable whereas the low one is never.*

Proof. See appendix A3. ■

Again, according to a continuity argument, we can extend the proposition to a non trivial open region of parameter space. Learnability conditions for the high BSE found above are a subset of the conditions for REE learnability (the symmetric case corresponds to the locus $x = 0.25$ in Fig. 2). This means that REE learnability is not global whenever the high BSE exists. In such cases (3.25) algorithms select between them depending on initial conditions.

One can doubt the economic relevance of BSEs, because most economic models, the reduced form of which encompasses (3.1), are restricted to $\beta < 1$ for several good economic reasons. As already noted, we set this question aside to simplify presentation of the basic dynamics. The interested reader is referred to Gaballo (2009) for an example of the economic relevance of BSEs in a very standard Lucas-type monetary model. There, non neutrality of the information channel from institutional forecasters and the private sector, combined with the use of constant gain learning algorithms, can trigger unpredictable endogenous and persistent switches to high volatility regimes (BSEs). In that context, the case in which agents have to learn about others' rationality *and* the exogenous process governing the actual law of motion is also analyzed.

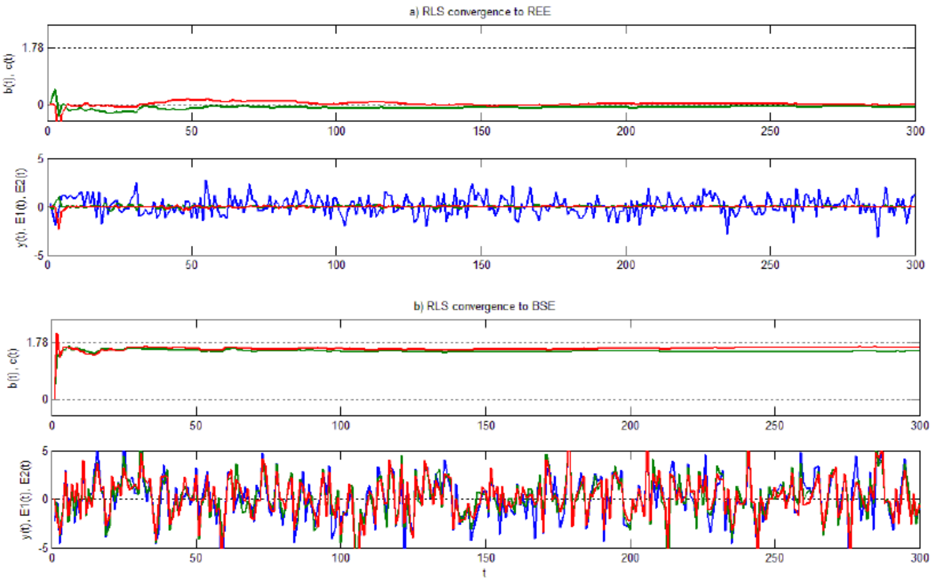

Figure 3.3: Two cases of RLS convergence up to 300 periods with $\lambda_1 = 0.5$, $\mathbf{x}_t = 0$, $\beta = 1.08$. Initial conditions are set equal for both at REE value $b_0 = c_0 = 0$. The two are run for different series of $v_{i,t-1}$ and η_t i.i.d. centred normally distributed shocks with unit variance.

3.3.3 Real Time Simulations

This section is devoted to real time simulation of the dynamic system at hand. The aim is to show convergence to learnable equilibria and how, in case of multiplicity, RLS generates path dependent selection between REE and high BSE.

Figure 3 shows two simulations of the system (3.18)-(3.25)-(3.26) up to 300 periods for the case of symmetrical agents ($\lambda_1 = 0.5$ and $\varepsilon_2 = \varepsilon_1 = 1$) with $\mathbf{x}_t = 0, \rho_v = 0$ and $\beta = 1.08$. The unique REE ($\widehat{b} = \widehat{c} = 0$) is $y_t^* = \eta_t$. The expression for the high BSE is obtained substituting BSE coefficient values $\widehat{b}_+ = \widehat{c}_+ = 1.788$ computed from (3.24), in (3.18). All shocks $\nu_{i,t}$, η_t are normally i.i.d. with unit variance. Initial conditions are set equal to the REE values $b_0 = c_0 = 0$ for both experiments.

Figure 3a represents the REE benchmark case. The first box shows evolution of estimated coefficients b_{t-1} and c_{t-1} driven by the RLS algorithm. In the experiment, b_{t-1} and c_{t-1} quickly converge to zero. The corresponding series of output gap and institutional forecasters expectations are shown in the second box. Expectations are closely rational expectation so they are denoted by a flat line at $y_t = 0$, whereas output gap exhibits stochastic fluctuations totally generated by η_t.

The experiment plotted in figure 3b is run with the same calibration and the same conventions as the one above, but with different shock series. The picture provides a numerical example of learnability of the high BSE. It gives a concrete intuition of how path-dependent selection among different learnable equilibria occurs in real time. The courses of estimated coefficients soon jump far away from the REE and converge towards the high BSE values. The output gap dynamics shown in the last box is dramatically different from the benchmark case. Firstly, expectations show substantial volatility that self-fulfills reciprocal agents' beliefs of irrational exuberance. Secondly, the overall variance of the process is greatly enhanced. The theoretical excess volatility is calculated from (3.19) and is about three, that is, the overall process has a theoretical variance of four.

3.4 Conclusions

Rationality of others is something that can theoretically be assessed in real time. The paper showed how it is possible to extend adaptive learning to a truly behavioural uncertainty problem in real time as eductive learning does in notional time. We investigated the simplest case of a first-order, self-referential model where private sector expectations are polarized by two institutional forecasters. In the simple self-referential model presented, interactions between institutional forecasters matter as long as every institutional forecaster has a non-negligible impact on the aggregate expectation. Therefore, to have consistent expectations, both institutional forecasters have to forecast the forecasts of the other.

To isolate the behavioural uncertainty problem, we assumed agents hold perfect knowledge of exogenous determinants of the economy but they do not know exactly what others expect. Since they are intrinsically heterogeneous, they cannot have pointwise knowledge of others' simultaneous expectations, but they make stochastic measurement errors in detecting other's simultaneous expectations. Institutional forecasters have to assess in real time if displacements of the actual output gap from the rational expectation are independent of observed displacements of other's simultaneous expectations from the rational expectation. If this is the case, institutional forecasters would no longer care about the noisy signal and would just forecast the rational expectation. Otherwise they would exhibit interdependent and excessively volatile expectations as they would consider the behavioral signal, so self-confirming reciprocal beliefs of irrational exuberance. Institutional forecasters have incentive to extract the signal of others' rationality because they fully recognize the self-referential nature of the economy. They are not boundedly rational as is generally assumed in adaptive learning.

The unique REE can arise from a prior disequilibrium when agents do not share any knowledge. Convergence to REE results from decentralized learning dynamics in which agents learn about the rationality of others in real time. Moreover, REE attainability is proved to be particularly robust to correlation and systematic bias in observational errors, but, *even if* institutional forecasters are perfectly informed about the economic fundamentals and acknowledge their own influence on the economy, there exists at least one learnable equilibrium different from the REE. Such equilibria are tagged behavioural sunspot equilibria. They originate because agents' observational errors can in some cases generate the self-sustaining and reciprocal belief of the other one being irrationally exuberant. These equilibria exhibit larger variance around the steady state determined by observational errors. In contrast to other sunspot equilibria and in line with the general approach, behavioral sunspot equilibria don't require any common knowledge property, but arise from decentralized coordination among non-cooperative agents. Moreover, the existence of BSEs is not alternative to REE, but they coexist in a large region of parameter space. In such a region real time learning dynamics selects among them.

Appendix

A1.

Spelling out condition (3.21a), we have

$$\beta\left(\frac{b\left(\lambda+(1-\lambda)c\right)}{\left(1-bc\right)^2}\varepsilon_1+\frac{c^2(\lambda b+(1-\lambda))}{\left(1-bc\right)^2}\varepsilon_2+\frac{c(\lambda b+(1-\lambda))+cb\left(\lambda+(1-\lambda)c\right)}{(1-bc)^2}\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+$$
$$-T_b\left(\frac{1}{\left(1-bc\right)^2}\varepsilon_1+\frac{c^2}{\left(1-bc\right)^2}\varepsilon_2+2\frac{c}{\left(1-bc\right)^2}\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)=0,$$

$$\beta\left(\frac{b\left(\lambda+(1-\lambda)c\right)}{\left(1-bc\right)^2}\left(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+\frac{c(\lambda b+(1-\lambda))}{\left(1-bc\right)^2}\left(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\right)+$$
$$-T_b\left(\frac{1}{\left(1-bc\right)^2}\varepsilon_1+\frac{c}{\left(1-bc\right)^2}\left(c\varepsilon_2+2\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)\right)=0.$$

Finally, the projected T-map for b and c (agent 2 is mirror-like) is given by

$$T_b=\beta\left(\frac{b\left(\lambda+(1-\lambda)c\right)\left(\varepsilon_1+c\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+c(\lambda b+(1-\lambda))\left(c\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_1+\varepsilon_2c^2+2c\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right),$$

$$T_c=\beta\left(\frac{c((1-\lambda)+\lambda b)\left(\varepsilon_2+b\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)+b\left((1-\lambda)c+\lambda\right)\left(b\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2}\rho_v\right)}{\varepsilon_2+\varepsilon_1b^2+2b\sqrt{\varepsilon_1\varepsilon_2}\rho_v}\right).$$

A2.

Let us define

$$K_{\left(\widehat{b},\widehat{c}\right)}\equiv JT(b,c)|_{\left(\widehat{b},\widehat{c}\right)}-\mathbf{I}_2 \tag{3.29}$$

where $\mathbf{I}_2$ is a 2×2 identity matrix. If $K_{\left(\widehat{b},\widehat{c}\right)}$ has negative eigenvalues then learnability of $\left(\widehat{b},\widehat{c}\right)$ holds. REE matrix K is given by

$$K_{(0,0)}=\begin{pmatrix}\beta\lambda-1 & \beta(1-\lambda)\frac{\sqrt{\varepsilon_1\varepsilon_2}\rho_v}{\varepsilon_1}\\ \beta\lambda\frac{\sqrt{\varepsilon_1\varepsilon_2}\rho_v}{\varepsilon_2} & \beta\left(1-\lambda\right)-1\end{pmatrix}. \tag{3.30}$$

To find conditions for which (3.30) has negative eigenvalues, it is enough to evaluate signs of the trace and the determinant. In particular, both eigenvalues are negative if and only if the trace (sum of eigenvalues) is negative and the determinant (product of eigenvalues) is positive. The trace of (3.30) in $(0,0)$, namely $\mathbf{T}(K_{(0,0)})$, is

$$\mathbf{T}(K_{(0,0)}) = \beta\lambda - 1 + \beta(1-\lambda) - 1 = \beta - 2,$$

so that $\mathbf{T}(K_{(0,0)}) < 0$ implies $\beta < 2$. The determinant $\mathbf{\Delta}(K_{(0,0)})$ takes the form

$$\mathbf{\Delta}(K_{(0,0)}) = (1-\rho_v^2)\lambda(1-\lambda)\beta^2 - \beta + 1,$$

and is not negative if and only if

$$\beta \leq \frac{1-\sqrt{1-4\lambda(1-\lambda)\left(1-\rho_v^2\right)}}{2\lambda(1-\lambda)\left(1-\rho_v^2\right)}, \quad \beta \geq \frac{1+\sqrt{1-4\lambda(1-\lambda)\left(1-\rho_v^2\right)}}{2\lambda(1-\lambda)\left(1-\rho_v^2\right)}, \tag{3.31}$$

since $\rho_v \in [-1,1]$ as a direct consequence of the Cauchy–Schwarz inequality. Finally, since

$$\frac{1-\sqrt{1-4\lambda(1-\lambda)\left(1-\rho_v^2\right)}}{2\lambda(1-\lambda)\left(1-\rho_v^2\right)} \leq 2 \leq \frac{1+\sqrt{1-4\lambda(1-\lambda)\left(1-\rho_v^2\right)}}{2\lambda(1-\lambda)\left(1-\rho_v^2\right)}$$

is always true, as $0 < 4\lambda(1-\lambda)\left(1-\rho_v^2\right) < 1$ is always true, we conclude that (3.27) is the ultimate condition.

A3.

Substituting $\lambda = 1/2$ and $\varepsilon_1 = \varepsilon_2$, matrix $K_{\left(\widehat{b},\widehat{c}\right)}$, becomes

$$K_{\left(\widehat{b},\widehat{c}\right)} = \begin{pmatrix} \frac{((\beta/2)-1)\widehat{b}^2+((\beta/2))\widehat{b}+(\beta/2)-1}{1+\widehat{b}^2} & \beta\frac{-\widehat{b}^3+3\widehat{b}}{2\left(1+\widehat{b}^2\right)^2} \\ \beta\frac{-\widehat{b}^3+3\widehat{b}}{2\left(1+\widehat{b}^2\right)^2} & \frac{((\beta/2)-1)\widehat{b}^2+(\beta/2)\widehat{b}+(\beta/2)-1}{1+\widehat{b}^2} \end{pmatrix} \tag{3.32}$$

where $\widehat{b} = \widehat{c}$. A certain equilibrium $\left(\widehat{b},\widehat{c}\right)$ is learnable if and only if matrix $K_{\left(\widehat{b},\widehat{c}\right)}$ has all negative eigenvalues.

To discuss the sign of eigenvalues of a 2×2 matrix, it is enough to study the signs of the trace and the determinant. The matrix at hand has all negative eigenvalues if and only if it has a negative trace (sum of eigenvalues is negative) and a positive determinant (product of eigenvalues is positive). We will discuss them separately in the case $\rho_v = 0$.

Trace The two entries on the main diagonal are equal, so, if the first entry is negative, the trace is negative too. The inequality is written as

$$((\beta/2)-1)\,\widehat{b}^2 + (\beta/2)\,\widehat{b} + (\beta/2) - 1 < 0 \tag{3.33}$$

the determinant of which is

$$\Delta = -3\left(\beta/2\right)^2 + 8\left(\beta/2\right) - 4$$

that is greater than zero only if $4/3 < \beta < 4$. Recalling that for existence of BSEs, $\beta \geq 1$, and since for $\beta > 2$ (3.33) is a convex curve, its solution is

$$\nexists\, \widehat{b} = \widehat{c} \ \text{ with } \beta > 4 \tag{3.34a}$$

$$s_- \leq \widehat{b} = \widehat{c} \leq s_+ \ \text{ with } 2 \leq \beta \leq 4 \tag{3.34b}$$

$$\widehat{b} = \widehat{c} \leq s_-, \ \ \widehat{b} = \widehat{c} \geq s_+ \ \text{ with } \frac{4}{3} \leq \beta < 2 \tag{3.34c}$$

$$\forall \widehat{b} = \widehat{c} \ \text{ with } 1 \leq \beta < \frac{4}{3} \tag{3.34d}$$

where

$$s_- = \frac{\beta - 2\sqrt{-3\left(\beta/2\right)^2 + 8\left(\beta/2\right) - 4}}{2-\beta},$$

$$s_+ = \frac{\beta + 2\sqrt{-3\left(\beta/2\right)^2 + 8\left(\beta/2\right) - 4}}{2-\beta}.$$

It can easily be checked[12] that the above conditions are always satisfied. We can conclude that (b_+, c_+) and (b_-, c_-) both exist and satisfy conditions (3.34c)-(3.34d), so that (3.33) holds in the range

$$\beta \in [1, 2) \tag{3.35}$$

that is, the range in which BSEs exist in the symmetric case. We use condition (3.35) to restrict analysis of the determinant.

Determinant The determinant of $K_{\left(\widehat{b},\widehat{c}\right)}$ is equal to $A^2 - B^2$ where

$$A = \frac{\left(\left(\beta/2\right) - 1\right)\widehat{b}^2 + \left(\beta/2\right)\widehat{b} + \left(\beta/2\right) - 1}{1 + \widehat{b}^2},$$

$$B = \frac{-\left(\beta/2\right)\widehat{b}^3 + 3\left(\beta/2\right)\widehat{b}}{\left(1 + \widehat{b}^2\right)^2}.$$

The condition to be studied is

$$A^2 - B^2 > 0, \tag{3.36}$$

[12] Exploit the fact that $-\left(3/4\right)\beta^2 + 4\beta - 4 < \beta - 1$ is always true.

that is true if and only if

$$|A| > |B|. \tag{3.37}$$

Learnability conditions require A to be negative, so that we only have to distinguish two cases with B negative or positive. Let us analyse the sign of the second entry. After some rearrangements the inequality is

$$-(\beta/2)\,\widehat{b}^3 + 3(\beta/2)\,\widehat{b} < 0. \tag{3.38}$$

Since $(\beta/2)$ is positive in the range (3.35), (3.38) is negative if and only if $\widehat{b} > \sqrt{3}$.

First case: if B is positive then (3.37) is

$$A < -B, \tag{3.39}$$

Let us go through the explicit analysis of (3.39)

$$\left(((\beta/2)-1)\,\widehat{b}^2 + (\beta/2)\,\widehat{b} + (\beta/2) - 1\right)\left(1+\widehat{b}^2\right) < (\beta/2)\,\widehat{b}^3 - 3(\beta/2)\,\widehat{b},$$

for $\widehat{b} = b_+ \in [1, \sqrt{3})$ and $\widehat{b} = b_- \in (0,1)$ where, $b_+$and b_- are given by (3.24).

We obtain

$$\left(\widehat{b}^4 + 2b^2 + \frac{4\beta}{\beta-2}\widehat{b} + 1\right)((\beta/2)-1) < 0, \tag{3.40}$$

and, since $((\beta/2)-1)$ is always negative in (3.35), (3.40) is satisfied whenever

$$\widehat{b}^4 + 2b^2 - \frac{4\beta}{2-\beta}\widehat{b} + 1 > 0. \tag{3.41}$$

Using the identity $4\beta/(2-\beta) = 2(b_+ + b_-)$ and rearranging, we express (3.41) as

$$\left(\widehat{b}^2 + 1\right)^2 > 2(b_+ + b_-)\,\widehat{b}.$$

Notice that $(b_+ + b_-)\,\widehat{b} = \left(\widehat{b}^2 + 1\right)$ in the two cases $\widehat{b} = b_+$ and $\widehat{b} = b_-$. Thus since $\widehat{b} > 0$ we can reduce latter inequality further to

$$\widehat{b}^2 + 1 > 2$$

that always holds for $\widehat{b} = b_+ \in [1, \sqrt{3})$ and never for $\widehat{b} = b_- \in (0,1)$.

Second case: if B is negative then (3.37) is

$$A < B. \tag{3.42}$$

Let us go through the explicit analysis of (3.42)

$$\left(\left((\beta/2)-1\right)\widehat{b}^2+(\beta/2)\,\widehat{b}+(\beta/2)-1\right)\left(1+\widehat{b}^2\right)<-\left(\beta/2\right)\widehat{b}^3+3\left(\beta/2\right)\widehat{b},$$

for $\widehat{b}=b_+\in[\sqrt{3},\infty)$.

We obtain

$$\left(b_+^4-\frac{2\beta}{2-\beta}b_+^3+2b_+^2+\frac{2\beta}{2-\beta}b_++1\right)\left(\left(\beta/2\right)-1\right)<0. \tag{3.43}$$

and, since $((\beta/2)-1)$ is always negative in the range (3.35), (3.43) is satisfied whenever

$$b_+^4-\frac{2\beta}{2-\beta}b_+^3+2b_+^2+\frac{2\beta}{2-\beta}b_++1>0.$$

As before, using $(b_++b_-)=2\beta/\left(2-\beta\right)$ and $(b_++b_-)\,b_+=\left(b_+^2+1\right)$, we have

$$b_+^4-\left(b_+^2+1\right)b_+^2+2b_+^2+\left(b_+^2+1\right)+1>0,$$

that finally reduces simply to

$$2b_+^2+2>0$$

that always holds trivially in the range (3.35) since $b_+>0$. This concludes the proof and the proposition follows from the union of the solutions of the two cases.

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Chapter 4

Endogenous Switches of Volatility Regimes

Abstract The importance of central bank communication policies and statistical learning in expectations formation have been recently emphasized. The present work merges and innovates basic ideas from both approaches in two respects. Firstly, we analyse a Lucas-type monetary model where private sector expectations are influenced by two, and not only one, institutional forecasters. Strategic motives takes place because the rational expectations equilibrium (REE) arises as solution of the simultaneous coordination game played by such big actors. Therefore, and this is a second novelty, both institutional forecasters have to learn not only about fundamentals but also about the rationality of the other's expectations. We show that the use of constant gain learning algorithms by institutional forecasters can give rise to endogenous, unpredictable and persistent switches in volatility regimes. Specifically, inflation dynamics can suddenly switches from the unique REE to a behavioral sunspot equilibrium and viceversa.

4.1 Introduction

4.1.1 Changes in volatility regimes

This paper aims to provide a stylized model on how unpredictable and endogenous changes of volatility regimes can arise mainly because agents fail to form expectations independently. Volatility is one among the most important sources of uncertainty. Generally the higher and more frequent are fluctuations in the economy the higher are costs paid in terms of insurance or financial fragility. One of the most challenging task for economists is to understand when and how an high volatility crisis triggers.

The issue of excess volatility has recently received attention by the profession with special regard for US time series evolution after the second word war onwards (figure below).

Figure 4.1: US inflation (percentage change of Production Prices Index) time series of last fifty years. Grey bars denote recession periods. The picture suggests different volatility regimes with no strict correlation between fluctuation amplitude and growth cycles.

Cogley and Sargent (2005a), Sims and Zha (2006), Primiceri (2005) have found several different drifting and volatility regimes. These studies seems to give little importance to quantitative effect of monetary policy. Sims and Zha (2006) notice that:

> *"...the work of Cogley and Sargent and Primiceri all fits with the notion that the data do not deliver clear evidence of parameter change unless one imposes strong, and potentially controversial, overidentifying assumptions."*.

Paying such price, Cogley and Sargent (2005b) and Primiceri (2006) gives an explanation volatility changes in terms of learning by the central bank. In their study there is evidence that high inflation in 70's would had risen because some years passed for the FED to correctly identify the model. In other words, Lucas' lesson would have been learned after evidence of it has been produced by the implementation of wrong policies. Nevertheless, even if partially in conflict, all these findings are in line with the suggestion coming from the picture above that high volatility periods and recessions are not significantly correlated. This would address the issue of volatility inflation changes to explanations not so strictly linked to cyclical real economy determinants.

4.1.2 Behavioral uncertainty, interdependent expectations and learning

An important part of the profession places now more and more emphasis on the role of central bank as focal point for agents' expectations, stressing the pre-eminence of the communication policy on the mere control of monetary determinants of the economy (a good introduction to the issue is Morris and Shin (2007)). Agents look at central bank expectations because everyone knows all others are looking at it, so that, central bank expectations provides noisy information on what the others are simultaneously expecting. This simple empirical fact tells economic theory that the idea individuals are able to hold rational expectations *independently* cannot hold. If this was the case, each one would simply have all relevant information and signals coming from the central bank would be just redundant.

Adaptive learning (Marcet and Sargent 1989, Evans and Honkapohja 2001) answers the need to design a more reasonable and dynamic theory of expectation formation in contrast with dogmatic acceptance of rational expectations hypothesis. The central idea is that agents act as econometricians. They form expectations according to a theory (a perceived law of motion) that is calibrated estimating recursively the impact of exogenous variables as data become available in real time. This more realistic way to think about expectations as a further dimension of the dynamics of the system introduces new issues as the use of misspecificated theories (Evans, Honkapohja and Sargent 1989, Sargent 1999, Evans and Honkapohja 2001) and evolutionary competition among alternative statistical predictors (Branch and Evans 2006, 2007, Guse 2005).

In this paper, both basic ideas, namely the one about the importance of the coordination role of central bank and the one about adaptive formation of expectations, are merged[1] and innovated in two respects to explain changes in volatility regimes. We analyse the setting in which, firstly, more than one, in this case two, institutional forecasters polarize private sector expectations and, secondly, professional forecasters use adaptive learning not only to learn about fundamentals but also to assess rationality of the other institutional forecaster's expectations.

Rating agencies, market leaders, fiscal authorities generally influence private sector expectations as well, and sometimes more, than the central bank. In general, whenever more than one agent has non-negligible impact on the aggregate expectation, holding

[1] Both arguments could have a further point of contact in the idea that estimation is a costly activity. The most part of agents cannot solve individually their own forecasting problem because the great amount of resources (at least cognitive) needed in gathering and processing all information to "produce" statistically consistent prediction. Therefore, agents look at central bank that maintains sufficient resources to form expectations according, in the best of cases, to an optimal statistical analysis of available data in light of the right theory underlying the working of the economy.

rational expectations is a best action if and only if all others do the same. Therefore, because behavioral uncertainty, agents have the incentive to understand how others' expectations affect the actual economic course. The interaction entailed by the coordination expectation game among institutional forecasters is a first source of expectations interdependence. A second one is entailed by the role of institutional forecasters acting as focal point for private sector's expectations. The latter is a one-way dependence linking private sector's beliefs to institutional forecasters' expectations whereas the former is a reciprocal interdependence among institutional forecasters' expectations.

In this paper, particular emphasis is placed on interaction between constant gain adaptive learning and these two forms of expectations interdependence being among primary reasons for persistent excess volatility triggering. To the aim we will simplify the setting in order to enlighten the basic mechanism and to make the analysis rigorous but handy. We don't want to neglect other determinants of excess volatility, but we aim to covey a first idea on how behavioral uncertainty alone can be enough to generate endogenous and unpredictable switches in volatility regimes.

4.1.3 Learning and communication

This work is a natural extension of Gaballo (2009). That paper investigates the learning dynamics of two institutional forecasters affected by behavioral uncertainty, but perfectly informed about both the exogenous determinants and the self-referential nature of the economy. Behavioral uncertainty arises in the sense they only have noisy observations of the simultaneous expectation of the other agent. Each agent estimates a coefficient weighting the noisy signal in expectation formation process responding to her own incentive to refine their forecasts. An equilibrium requires expectations to be locally optimal linear projections given behavioral uncertainty restrictions on the information set. Rational expectations equilibrium occurs whenever the estimated coefficient is zero, so that the behavioral noisy information is discarded. Otherwise, because the endogenous and interactive working of the learning algorithm, some equilibria different from REE can arise entailing excess volatility regimes. Those equilibria have been tagged behavioral sunspots equilibria (BSE). They configure as a coordination failure in that both agents use irrelevant information.

In the present study, this scheme governs the arising of endogenous interdependence among institutional forecasters' expectations and it is implemented in a simple monetary Lucas-type model (the same of Evans and Branch (2006)). The microfoundation of the model is presented in the first section, nevertheless the results are not linked to the particular specification at hand. There are no novelties in the model in se. The setting is very simple but it obeys to general economic incentives and constraints such as transversality conditions and non negativity of prices. This is enough to deal with one among the primary concerns of this paper, that is, to defend, in principle, economic relevance of BSE.

Four new directions are explored. First, institutional forecasters learn not only about others' rationality but *also* about the fundamentals. This way it is showed how to merge the theme of learning about others' rationality with the classical theme of learning about fundamentals already developed in standard adaptive learning literature. We will refer to these two connected learning dynamics as the learning determinants of inflation dynamics. Second, the transmission channel from institutional forecasters to private sector is not neutral in that institutional forecasters' estimates are imitated with a noisy, possibly correlated, perturbation. This feature adds a truly macroeconomic flavour in that it reconciles the classical "forecasting the forecast of others' problem", where agents have non negligible impact on aggregate outcomes, with a non-trivial general equilibrium perspective, where an ocean of negligible agents are assumed. In other words, the problem faced by institutional forecasters is not their mere expectation coordination problem because the non-neutrality of the information channel from them to the ocean of agents forming the private sector possibly alters the feedback mechanism. This constitutes what we will call the communication determinant of inflation dynamics. Third, institutional forecasters use a constant gain learning algorithm instead of recursive ordinary least square. Differently from recursive ordinary least square this rule is time invariant and it allows for persistent learning. This work provides also an example on how constant gain cannot only learn a structural change, but it can also trigger it endogenously. Finally, the paper extends analysis of BSE learneability to the case of correlation between institutional forecasters' observational errors. This is a natural extension since institutional forecasters are part of the same public environment, so that they are influenced by same factors. In other words, it is likely, in some extent, that both either have pessimistic perceptions of the other's expectations or have optimistic ones.

4.1.4 Switching from REE to BSE and viceversa

The main result of the paper is to provide a simple model that exhibit very standard rational expectations behavior *and,* at the same time, it has the potentiality to trigger persistent and endogenous changes in volatility without relying on any Markov switching or additional aggregate shock. All this is basically due to the endogenization of agents' beliefs coordination as described in Gaballo (2009). Here the basic mechanism is implemented and further developed in the context of a simple monetary model in order to defend, in principle, economic relevance of BSE.

The rational expectations equilibrium learneability is proved to generally hold and to be particularly robust to correlation in observational errors. Nevertheless, *even if* institutional forecasters successfully learn about economic fundamentals and rationality of others, one learnable BSE may suddenly arise. Conditions for emergence of a learneable BSE are fulfilled if the transmission channel from institutional forecaster to private sector causes even a very little average amplification of the signal passed by institutional forecasters. The striking feature of this model is that BSE is not alternative to rational expectation equilibrium (REE), but they coexist in a large region of the parameter space. In such a region, real time constant gain learning dynamics selects among them and endogenous and unpredictable switches from one equilibrium to the other can generally arise. I show with numerical simulations how a structural switch from the rational expectations equilibrium to BSE may occur endogenously. Persistent deviations from REE result because, even if agents are all rational and able to consistently estimate fundamentals, they may fail to extract the signal of others' rationality, falling into a coordination failure trap.

4.1.5 Related literature on excess volatility

Branch and Evans (2007) consider the theme of learning but they focus on evolutionary competition among expectation formation theories in a simple self referential model. Even if central bank is typically the most authoritative forecasting institution several different theories of the same economy are actually employed by agents to forecast. In an evolutionary contest competing theories can coexist because no one is able to perform better than others given their distribution over the population. Using a Lucas-type monetary model, Branch and Evans shape such environment in which different underparameterized theories are available to agents that choose among them on the basis of past performance. They show that Misspecification Equilibria (Branch and Evans 2006a) can arise giving rise to persistent stochastic volatility. Nevertheless, there stochastic volatility is permanent and finally relies on the unavailability of a correctly specified predictor, the only one potentially consistent with REE. This is not the case for the model we are going to present since excess volatility regimes and REE regimes alternates via an endogenous mechanism.

Related is also an extensive stream of literature on excess volatility in asset market returns. We can distinguish mainly four approaches in Macroeconomics. First, Timmermann (1993, 1996), Brennan and Xia (2001) and Cogley and Sargent (2006) among others assume agents implementing Bayesian learning on the dividend process. Those models are not self-referential nature, since agents beliefs do not influence the market outcomes. It is common sense and a simple empirical exercise to test that financial operators actually react to changes in prices as well. Differently, Carceles-Poveda and Giannitsarou

(2008), Adam, Marcet and Nicolini (2008a) and Bullard, Evans and Honkapohja (2007) properly takes in to account agents adaptively learning about the prices level. As clarified by Adam, Marcet and Nicolini (2008b) learning about the price level is justified by uncertainty on the marginal agents' expectations, therefore, this scheme considers implicitly the self-referential nature of the model. Later works building on Brock and Hommes (1997, 1998) assumes agents choose among a set of very few sophisticated predictors of the price level relying on relative past performance. Such setting can give rise to complex dynamics and strange attractors. Finally, a recent approach initiated by Allen, Morris and Shin (2006) focuses on the role of high order beliefs of rational short lived agents.

The most important feature of the proposed model in front of quoted literature is that the model is consistent and not alternative to REE. In other words, models above rely on some mechanism that either is exogenously imposed at an aggregate level or persistently alters the volatility regime of the dynamics. Differently, in the model presented below, persistent high volatility regimes endogenously (and unpredictably) arise from a REE regime and viceversa. Moreover the extra noise possibly entering in the equilibrium solution is justified at a micro level, that is, it is not an arbitrary aggregate shock.

4.2 Model

4.2.1 A Lucas-type economy

The primary concern of this section is to provide a simple fully microfounded model with the aim to defend, in principle, the economic relevance of behavioral sunspot equilibria. Of course the choice is functional to the scope, so the model is rich enough to embodies standard economic incentives and constraints usually assumed, but also simple enough to have an handy reduced form. Specifically, we will derive a simple Lucas-type monetary model where expectations of current inflation influences actual inflation. It is not a task of this paper to introduce novelties concerning the model in se. To make easier the comparison with closest literature, we will assume the same model with slightly different notation as in Branch and Evans (2007). The key assumptions are the following. We use the convention of a yeoman farmer model (as in Woodford (2003)) provided with a money-in-the-utility function. This is enough to generate a non trivial demand for money responding to classical quantity theory of money without referring to any specification of the financial market. Nevertheless, as first approximation, such demand is assumed to be interest inelastic, so that, dependence of higher order beliefs (forward expectations) is avoided. Finally, we assume a fraction of firms have to set quantities a period before. The latter hypothesis makes expectations about current inflation matter. We now detail the model.

Households. Each farmer produces a differentiated good and sells it in a monopolistically competitive market. In order to introduce price stickiness it is enough to allow for endogenous goods supply. Technology for a representative firm belonging to industry i is given by the following

$$Y_{it} = \psi_t \Omega_t^{-1/(1+\eta)} N_{it}$$

where Ω_{t-1} is the unit labor requirement, ψ_t is a stochastic disturbance and N_{it} is the quantity of labour specific for industry i employed at time t. Let's assume two types of industries; extension to an arbitrary number is straightforward. A representative households solves

$$\max_{\{C_{it},M_{it},N_{it},B_{it}\}} E_0^i \sum_{t=0}^{\infty} \beta^t \frac{C_{it}^{1-\gamma} + M_{it}^{1-\gamma}}{1-\gamma} - \frac{N_{it}^{1+\eta}}{1+\eta}$$
$$\text{s.t.} \quad C_{it} + M_{it} + B_{it} = Y_{it} + \frac{P_{t-1}}{P_t} M_{it-1} + \frac{P_{t-1}}{P_t} (1 + i_{t-1}) B_{it-1}$$

where i_t is the nominal one-period interest rate on debt, E^i is conditional expectation given agent i's information set, B_{it} and M_{it} are respectively bond stock and nominal stock of money held by agent i at time t, and

$$C_i = \left(\int C_{i,j}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}},$$
$$P_t = \left(\int P_{j,t}^{1-\theta} dj \right)^{\frac{1}{1-\theta}},$$

are CES indexes with $C_{i,j}$ and $P_{j,t}$ being respectively consumption of good j by agent i and price of good j. The aggregate demand Y_t is equal to the integral of individual cost-minimizing demand over agents and goods, formally

$$Y_t = \int \int \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} n_i C_{it} \, di \, dj = \int Y_{it} \, di = \int \left(\frac{P_{j,t}}{P_t} \right)^{-\theta} Y_{j,t} \, dj$$

where Y_{it}, Y_{jt} and n_i are respectively individual aggregate demand over goods j, aggregate demand of good j over individuals i and the fraction of firm type i. The household's first-order conditions can be written as,

$$\begin{aligned}
C &: \beta^t C_{it}^{-\gamma} - \lambda_{it} = 0 \\
M &: \beta^t M_{it}^{-\gamma} - \lambda_{it} + E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\
B &: -\lambda_{it} + (1 + i_t) E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\
N &: -\beta^t N_{it}^{\eta} + \lambda_{it} \frac{Y_{it}}{N_{it}} = 0
\end{aligned}$$

where λ_{it} is the Lagrangian multiplier for the budget constraint. We can rewrite condition above solving for λ_{it}. We obtain

$$\begin{aligned}
C_{it}^{-\gamma} &= \Omega_{it-1} Y_{it}^{\eta}, \\
C_{it}^{-\gamma} &= M_{it}^{-\gamma} + \beta E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}}, \\
C_{it}^{-\gamma} &= \beta(1 + i_t) E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}}.
\end{aligned}$$

These conditions must be satisfied for all i and in all t . In the steady-state $P_{t+1}/P_t = 1$ and $\beta(1 + i_t) = 1$. Combining Euler equations above it is possible to solve for the money-demand function:

$$M_{it} = \left(\frac{i_t}{1 + i_t} \right)^{-\frac{1}{\gamma}} C_{it}.$$

Following Walsh (2003), let $\gamma \to \infty$, so that money demand is interest inelastic and the equilibrium in the money-market requires

$$M_t = Y_t$$

and taking logs of both sides we derive a simple version of the well-known quantity theory of money

$$\ln \overline{M}_t - \ln P_t = \ln Y_t \ , \tag{4.1}$$

where $\overline{M}_t$ is real money supply. The latter represents the aggregate demand (AD) equation. Notice also such assumption makes consumption and money demand to be independent from (heterogeneous) expectations on future inflation rate. The latter represents the aggregate demand (AD) equation.

Production. Firms set price to maximize profits. Let $P_{i,t}$ be the price in industry i settled by firms taking as given the aggregate price-index P_t. Then a firm's profit function is

$$\Pi \equiv (P_{i,t} - P_t)\, Y_{i,t} = P_{i,t} Y_{i,t} - \frac{\psi_t \Omega_{t-1} Y_{i,t}^{1+\eta} P_t}{C_{i,t}^{-\gamma}}.$$

whose F.O.C. is,

$$\frac{\partial \Pi}{\partial P_{i,t}} + \frac{\partial \Pi}{\partial Y_{i,t}} \frac{\partial Y_{i,t}}{\partial P_{i,t}} = Y_{i,t} + \left(-\theta Y_t \left(\frac{P_t}{P_{i,t}}\right)^{\theta}\right) \left(P_{i,t} - \frac{\psi_t \Omega_{t-1} (1+\eta)\, Y_{i,t}^{\eta} P_t}{C_{i,t}^{-\gamma}}\right) = 0$$

that reduces to

$$\left(\frac{P_{i,t}}{P_t}\right)^{1+\theta\eta} = \frac{\theta}{\theta - 1} \frac{\psi_t \Omega_{t-1} Y_t^{\eta}}{C_{i,t}^{-\gamma}},$$

or, in log form

$$\ln(P_{i,t}) = \ln(P_t) + \frac{\eta}{1+\theta\eta} \ln(Y_t) - \frac{\gamma}{1+\theta\eta} \ln(C_{i,t}) + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \ln\left(\frac{\theta}{\theta-1}\psi_t\right).$$

Following Woodford, assume there is a fraction τ of firms that set prices optimally in every period, while the remaining set their prices one period in advance. Denote $P_{i,f}$,$P_{i,d}$ as the prices of an industry i respectively of type f with flexible prices and d with predetermined prices. Then the log-linearized pricing equations are:

$$\begin{aligned} \ln P_{i,f,t} &= \ln(P_t) + \frac{\eta}{1+\theta\eta} \ln(Y_t) - \frac{\gamma}{1+\theta\eta} \ln(C_{i,t}) + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \zeta_t \ , \\ \ln P_{i,d,t} &= E_{t-1}^{d} \ln P_{i,f,t} \ . \end{aligned}$$

where ζ_t collects the stochastic term in ψ_t. With all agents types evenly distributed across industries it follows that the aggregate price-index can be approximated as,

$$\ln P_t = \tau\,(n \ln P_{1ft} + (1-n) \ln P_{2ft}) + (1-\tau)\left(n E_{t-1}^{1} \ln P_{1ft} + (1-n)\, E_{t-1}^{2} \ln P_{2ft}\right)$$

and since[2] $E_{t-1}^{i} \ln P_{ft} = E_{t-1}^{i} \ln P_t$ we have

$$\begin{aligned} \ln P_t - E_{t-1} \ln P_t &= \frac{\tau}{1-\tau}\,(n \ln P_{1ft} + (1-n) \ln P_{2ft} - \ln P_t) \\ &= \frac{\tau}{1-\tau}\left(\left(\frac{\eta-\gamma}{\theta}\right) \ln Y_t + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \zeta_t\right) \end{aligned}$$

Therefore, we have the aggregate supply (AS) relation

$$q_t \equiv \ln Y_t - \ln \Omega_{t-1} = \varphi_1 (\ln P_t - E_{t-1} \ln P_t) + \varphi_2 \ln \Omega_{t-1} + \varphi_3 \zeta_t$$

[2] Consider $E_{t-1}^{i} \ln p_t = \tau E_{t-1}^{i} \ln p_{ft} + (1-\tau)\, E_{t-1}^{i} \ln p_{ft}$.

with

$$\varphi_1 = \frac{\theta(1-\tau)}{\tau(\eta-\gamma)}, \quad \varphi_2 = -\frac{1+\tau\theta\eta}{\tau+\tau\theta\eta}, \quad \varphi_3 = -\frac{(1-\tau)}{\tau},$$

where, for example, $\ln \Omega_{t-1}$ follows a deterministic trend. The AS is a kind of new classical Phillips curve encompassing the one in Lucas (1973), Kydland and Prescott (1977), Sargent (1999), and Woodford (2003).

The economy is represented by equations for aggregate supply (AS) and aggregate demand (AD):

$$\begin{aligned} AS &: \; q_t = \varphi_1(p_t - p_t^e) + \varphi_2\omega_{t-1} + \varphi_3\zeta_t \\ AD &: \; q_t = m_t - p_t \end{aligned}$$

where p_t is the log of the price level, p_t^e is the log of expected price formed in $t-1$, m_t is the log of the money supply, q_t is the deviation of the log of real GDP from trend, ζ_t is an i.i.d. zero-mean shock, and ω_t is the log of the unit labor requirement.

Monetary Authority. Assume that the money supply follows

$$m_t - p_t = -(1+\xi)(p_t - p_{t-1}) + \delta\omega_{t-1} + u_t \quad \text{with} \quad \xi \geq 0,$$

where u_t is a white noise money supply shock. We are assuming that central bank can observe both p_t and y_t as in Sargent (1987) and Evans and Ramey (2006). Denoting $\pi_t = p_t - p_{t-1}$ we can write the law of motion for the economy in its expectations augmented Phillips curve form

$$\pi_t = \frac{\varphi_1}{1+\varphi_1+\xi}\pi_t^e + \frac{\varphi_2-\delta}{1+\varphi_1+\xi}\omega_{t-1} + \frac{\varphi_3}{1+\varphi_1+\xi}\zeta_t - u_t$$

or

$$\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\pi_t^e + \nu_t \tag{4.2}$$

where $\mathbf{z}'_{t-1} \equiv [1 \;\, \omega_{t-1}]$, $\beta = \frac{\varphi}{1+\varphi+\xi}$, $\boldsymbol{\alpha}' = \begin{bmatrix} 0 & \frac{\varphi_2-\delta}{1+\varphi+\xi} \end{bmatrix}$, $\nu_t \equiv \frac{\varphi_3}{1+\varphi_1+\xi}\zeta_t - u_t$. Note, in particular, that $0 \leq \beta < 1$. The reduced form of this Lucas-type model is very close to the cobweb one. The difference between the two is in the sign of the feedback form expectations: the latter entails a negative feedback, the former a positive one. Differently from the new-Keynesian framework, inflation at time t is affected by expectations at time $t-1$ instead that simultaneous expectations.

Equilibrium. A rational expectations equilibrium (REE) is a stationary sequence $\{\pi_t\}$ which is a solution to (4.2) given $\pi_t^e = E_{t-1}\pi_t$, where E_t is the conditional expectations operator. It is well known that (4.2) has a unique REE and that it is of the form

$$\overline{\pi}_t = (1-\beta)^{-1} \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \nu_t. \tag{4.3}$$

The REE is a stationary process and cannot explain volatility switching empirically observed. This paper will deal with such unsatisfactory property modifying expectation formation process. As it will be clear later, the reinforcement effect of agents' expectations is a necessary but not sufficient feature for our purposes. Specifically, provided the expectations feedback effect is always damping, the information transmission in the economy will play a key role for emergence of endogenous volatility regime switching. The following section will detail how we are going to modify the rational expectations hypothesis.

4.2.2 Expectations formation and information diffusion

This section aims to describe how aggregate expectation forms and evolves in time. We will introduce two essential hypothesis in place of the rational expectation hypothesis. First, non-trivial behavioral uncertainty takes place, second, private sector expectations are polarized by two institutional forecasters.

Non-trivial behavioral uncertainty. A general way to model behavioral uncertainty is to assume agents suffer a measurement error in detecting others' simultaneous expectations. Let's denote by $E^i_{t-1}(\cdot)$ agent i's expectation on $(\cdot)$ at time $t-1$. Behavioral uncertainty is entailed formally by

$$\left(E^i_{t-1}E^j_{t-1}\pi_t - E^j_{t-1}\pi_t\right) \equiv v_{i,t-1} \sim \Upsilon\left(0,\delta\right) \tag{4.4}$$

where $\nu_{i,t-1}$ is a stochastic measurement error drawn from a generic centred distribution function $\Upsilon_i\left(0,\delta\right)$ with zero mean and finite variance δ. In words, agents noisily perceive others' expectation. This is a first (reasonable) departure from REE paradigm in that, form a strict microfounded point of view, REE holds given common knowledge of every agent holds rational expectations. One may want to keep also non-centred distribution, or different type of distributions over the population. This has a sense and it can generate interesting dynamics, nevertheless it doesn't add nothing substantial,but some complexity , to the aim of this paper. As first exercise, we will focus on a unique centred distribution equal for every agent.

Behavioral uncertainty about others' expectations is at the basis of the "forecasting the forecast of others problem" originally posed by Townsend (1983). A following stream of literature investigates how agents can coordinate on rational expectations from a prior disequilibrium (Marcet and Sargent (1989), Sargent (1991), Singleton (1987), Kasa (2000) and Pearlman and Sargent (2004)). All these works consider explicitly a finite number of agents who form expectations independently. Nevertheless, in a general equilibrium perspective, to which the concept of REE refers to, the behavioral uncertainty problem can be just trivial as long as independent idiosyncratic deviations from the rational expectation vanish in the aggregation of an infinite number of agents. As long as individual deviations from the rational expectation are truly random and population is large enough, behavioral uncertainty doesn't add nothing substantial to the individual forecasting problem. In this sense, behavioral uncertainty is non-trivial as long as agents' deviations from REE prescriptions are driven by a common factor whose identification is crucial to optimally solve the individual forecasting problem. In fact, a reasonable doubt that there could exist a non trivial part of agents deviating in a correlated way from REE prescriptions would in turn justifies an individual rational departure from REE prescriptions.

Typically, the coordination of expectations on a particular deviation is yield by the introduction of an exogenous variable working as sunspot. Nevertheless, non-trivial behavioural uncertainty is not consistent with this idea because the emergence of a particular sunspot solution typically requires common knowledge that agents simultaneously believe in such solution. For that no behavioral uncertainty is actually involved in the classical definition of sunspots solutions. Differently, here we want to link the possibility of correlated deviations from REE prescriptions to the fact agents doubt that non-trivial behavioral uncertainty might takes place. To this aim expectation polarization hypothesis is introduced.

Expectations polarization. Let's start from the idea, consistent with statistical learning approach, that forming expectations is a costly activity at least form a cognitive point of view. It is unreasonable to assume that the most part of agents are expert in economics. It is more natural to sooner think they don't have a particular theory on how the economy works. Rather they rely on expectations of some more informed agent like a market leader, or a financial institution that has organizational skills and adequate resources to gather and rationally analyse information. Few institutional forecasters act as focal points for private sector expectations because economies of scale in the "production" of information are typically much stronger than in the production of any other good. The very small number of rating agencies in financial markets is an immediate example of this idea in real economy. In this sense institutional forecasters polarize public expectations.

Let' define formally the structural heterogeneity between an institutional forecasters and private sector. For the sake of simplicity assume there are only two institutional forecasters forming expectations according to

$$E^i_{t-1}x_t \equiv \mathbf{E}[x_t|\Omega^i_{t-1}], \quad \forall i = 1, 2. \tag{4.5}$$

In words institutional forecasters maintain mathematical expectation of the generic process x_t conditioned to available information up to time $t-1$. Assumption (4.5) is a formal specification of procedural rationality. It is natural to think professional forecasters are very few because information processing presents strong scale economy effects. We postpone the precise definition of Ω^i_{t-1} until the definition of their learning problem.

Differently, the private sector have the following expectation function specification

$$E^z_{t-1}\pi_t = E^{i_z}_{t-1}\pi_t + v_{z,t-1}, \tag{4.6}$$

where $E^z_{t-1}(\cdot)$ is nothing else then an imitation correspondence and i_z is the institutional forecaster noisily imitated by agent z belonging to private sector agents set $Z \equiv (0,1)$. Notice that the noise occurs since behavioral uncertainty assumption. If this working hypothesis is reasonable, agents expectations are polarized around few institutional forecasters' forecasts.

Therefore the aggregate expectation is

$$E_{t-1}\pi_t = \int\limits_{z\in Z} E^z_{t-1}\pi_t \; dz = \sum_{i=1,2} \lambda_i E^i_{t-1}\pi_t + \int\limits_{z\in Z} v_{z,t-1} \; dz \ , \quad \sum_{i=1,2} \lambda_i = 1 \tag{4.7}$$

where $\lambda_i \in (0,1)$ represents the size of the public relying on agent i's expectation. In the present work λ is an exogenous parameter. The extent of agent i's basin of audience, represented by λ_i, measures the average impact of agent i's expectation on the aggregate expectations. It would be very interesting to endogenize it with respect the relative performance of institutional forecasters. This route will be not undertaken in the present work. Nevertheless from here onward we focus on the case of two institutional forecasters polarizing evenly private sector ($\lambda_i = 1/2$). This assumption has a sense given observational error are equal and institutional forecasters both face the same problem, so that $\lambda_i = 1/2$ is by sure a rest point of the replication dynamics driven by institutional forecasters' relative performance.

Equation (4.7) invalidates the negligibility of agents individual impact in the economy as assumed in general equilibrium perspective. In particular, the impact of each institutional forecaster in the economy is equal and amounts half of the overall aggregate expectation effect. As long as expectations are strongly polarized strategic interaction motives arise among institutional forecasters in expectations formation. We are assuming each agent relies on institutional forecaster's expectations to form his own expectations. Therefore, noisy perceptions about the institutional forecasters' expectations provide information on a common factor embodied in agents' expectations that identifies eventual correlated deviations from REE.

Information diffusion. Figure 1 displays the information diffusion scheme entailed by assumptions above. Two institutional forecasters (red points) affect aggregate expectation calculated over an ocean of agents according to their respective audiences supposed to be equal. The aggregate expectation yields an actual inflation level as implied by (4.2). Moreover both institutional forecasters have noisy perceptions of the other institutional forecaster's simultaneous expectations. Arrows show flows of information. The two institutional forecasters analyse available data with statistical tools and, on the basis of their estimates, form expectation on future actual inflation.

Three are the key coefficients of the model: β is the feedback of aggregate expectation of current inflation on actual inflation, ρ_v is the correlation coefficient between institutional forecasters' observational errors, and finally γ denotes the covariance between institutional forecasters' expectations and the individual observational error committed by the private sector. The latter measures the non-neutrality of information channel and will be conveniently defined later. In sum, inflation dynamics is affected by learning determinants, that is, how institutional forecasters' expectations evolve in time, and communications determinants, that is what happens to information during the transmission from institutional forecasters to private sector. We will see soon in next section how those three parameters are enough to grasp basic phenomena arising from the interaction of learning (about fundamentals and rationality of others) and institutional communication.

4.3 From perceived to actual law of motion

4.3.1 Learning determinants

The emergence of the unique REE depends on the game played by institutional forecasters. Given the power of each institutional forecaster to displace actual inflation away from fundamentals, holding rational expectation is a best expectation if and only if each institutional forecasters believe the other one hold rational expectations. In order to satisfy this requirement institutional forecasters have a double task: learning about fundamentals and learning whether or not the other institutional forecaster, and hence a non trivial part of agents, has rational expectations.

Learning about fundamentals: the exogenous long-run component. The REE inflation rate (or fundamental inflation rate) is the long-run component of inflation, denoted by $\overline{\pi}_t$, determined by truly exogenous components. This is the only process compatible with long run equilibrium of agents' forecasts, that is, with rational expectations. Institutional forecasters learn about the fundamental inflation rate regressing a constant and the relevant exogenous variables affecting the economy on actual inflation, namely,

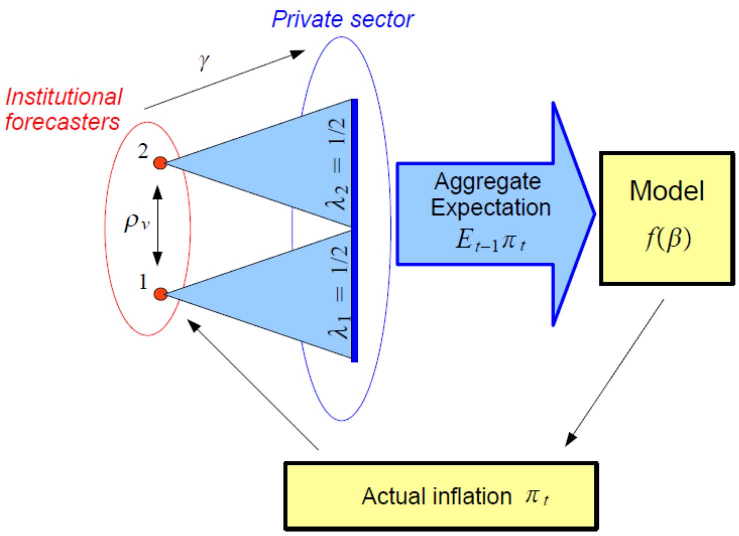

Figure 4.2: Information diffusion in the economy. Institutional forecasters 1 and 2 analyse data as they become available in time and produce statistically optimal forecasts. Institutional forecasters' expectations polarize evenly private sector expectations. The latter determine, jointly with other exogenous determinants, the actual inflation.

in our case, respectively $\mathbf{z}_{t-1}$ on inflation π_t. As standard in adaptive learning literature we assume they hold a correct perceived law of motions encompassing REE form

$$E^i_{t-1}\overline{\pi}_t = \mathbf{a}'_{i,t-1}\mathbf{z}_{t-1}$$

where $\mathbf{a}'_{i,t-1} \equiv [a^c_{i,t-1} \ a^{\omega}_{i,t-1}]$ are estimated coefficients. Specifically we assume $\mathbf{a}'_{i,t-1}$ is updated recursively in time according to the following constant stochastic gradient (CSG) rule

$$\mathbf{a}_{i,t-1} = \mathbf{a}_{i,t-2} + g_f \ \mathbf{z}_{t-2}\left(\pi_{t-1} - \mathbf{a}'_{i,t-2}\mathbf{z}_{t-2}\right), \tag{4.8}$$

where g_f is a constant gain smaller than one. Since same information is used, the two estimates asymptotically coincide irrespective of possibly different initial priors, that is $\lim_{t\to\infty}\mathbf{a}_{1,t-1} = \lim_{t\to\infty}\mathbf{a}_{2,t-1}$. Therefore, both have the same forecast of fundamental inflation, that we label $\overline{\pi}^e_t$, with approximation vanishing very soon.

Algorithm (4.8) is similar to the recursive version of OLS where the estimated correlation matrix is settled equal to the identity matrix and the gain coefficient is fixed[3]. CSG

[3] CSG is obtained form recursive constant gain OLS formula

$$\begin{aligned}\mathbf{a}_{t-1} &= \mathbf{a}_{t-2} + \overline{g}R^{-1}_{t-1}\mathbf{z}_{t-2}(\pi_{t-1} - \mathbf{a}'_{i,t-2}\mathbf{z}_{t-2})\\ R_{t-1} &= R_{t-2} + \overline{g}\left(\mathbf{z}_{t-2}\mathbf{z}'_{t-2} - R_{t-2}\right)\end{aligned}$$

converges to an ergodic distribution centred on the fixed point of the T-map whenever recursive OLS asymptotically converges (to a point). CSG, as any constant gain learning rule, exhibits permanent learning since more weight is given to more recent data. This makes these class of algorithms particularly suitable for learning structural changes. Recursive OLS on the contrary converges at the cost of a huge stickiness of the dynamics after relatively few repetitions. Moreover the CSG algorithm are also derived as optimal solution to a forecast errors variance minimization problem provided agents are "sensitive" to risk in a particular form. For details see Evans, Honkapohja and Williams (2005).

Learning about others' rationality: the endogenous idiosyncratic component. Even in case institutional forecasters correctly estimate fundamental inflation, actual inflation differs from the fundamental one at least for the exogenous stochastic noise ν_t. Nevertheless, because non-trivial behavioral uncertainty is in play, institutional forecasters cannot exclude that such stochastic deviations are due to idiosyncratic departures of aggregate expectation from the rational one. In particular, both institutional forecasters have to understand whether or not deviations from the REE are due to deviations of the other institutional forecasters' expectations from the rational expectation. In other words, they have to assess if the signal about others' expectations is informative about such departures. So, they estimate the optimal weight to give to noisy observations in order to refine their forecasts on actual idiosyncratic inflation deviations from the fundamental. If successful, they extract the signal of others' rationality in real time in that they assess that the noisy information about others' expectations is irrelevant.

They forecast idiosyncratic departure from the fundamental forecasted inflation $\overline{\pi}_t^e$ according to the rule

$$E_{t-1}^1 (\pi_t - \overline{\pi}_t^e) = b_{t-1} \left(E_{t-1}^2 \pi_t + \upsilon_{1,t-1} - \overline{\pi}_t^e \right), \tag{4.9a}$$
$$E_{t-1}^2 (\pi_t - \overline{\pi}_t^e) = c_{t-1} \left(E_{t-1}^1 \pi_t + \upsilon_{2,t-1} - \overline{\pi}_t^e \right), \tag{4.9b}$$

where b_{t-1} and c_{t-1} are recursively estimated with CSG

$$\begin{aligned} b_{t-1} &= b_{t-2} + g_d \left(E_{t-1}^2 \pi_t + \upsilon_{1,t-1} - \overline{\pi}_t^e \right) (\pi_{t-1} - E_{t-2}^1 \pi_{t-1}), \\ c_{t-1} &= c_{t-2} + g_d \left(E_{t-1}^1 \pi_t + \upsilon_{2,t-1} - \overline{\pi}_t^e \right) (\pi_{t-1} - E_{t-2}^2 \pi_{t-1}), \end{aligned}$$

where g_d is the updating gain, $(\pi_t - E_{t-2}^i \pi_t)$ is the forecast error and $\left(E_{t-1}^j \pi_t + \upsilon_{i,t-1} - \overline{\pi}_t^e \right)$ is the noisy observed displacements of others' expectations from the estimated fundamental one.

fixing $R_{t-1} = \mathbf{1}$. Therefore, in order to obtain adjustments comparable with constant gain OLS the gain has to be rescaled so that $g = \overline{g}/var(\mathbf{z}_t)$ given $\lim_{t\to\infty} R_{t-1} = var(z_t)$.

If (b_{t-1}, c_{t-1}) asymptotically converge to zero institutional forecasters will forecast the fundamental value, so that aggregate expectation will be a rational expectations. In other words, if all institutional forecasters are rational they would not need to condition their expectations on noisy observations of the other one's simultaneous expectations. But if this is not the case considering noisy observations actually improves the accuracy of forecasts. Learning is valuable exactly because this form of behavioral uncertainty is introduced. In this case, CSG has the advantages of showing convergence to equilibria and, at the same time, the possibility of endogenous and unpredictable shifts from the REE to a BSE. For details about this learning scheme the interested reader can refer to Gaballo (2009).

4.3.2 Communication determinants

Non-neutrality of the information channel. The process $v_{z,t-1}$ features the effect of information transmission from institutional forecasters to private sector, formally measured by $\int_{z\in Z} v_{z,t-1}\ dz$ in (4.7). It is convenient to express $v_{z,t-1}$ in the following way

$$v_{z,t} = \gamma(E^{i_z}_{t-1}\pi_t - \overline{\pi}_t) + (1-\gamma)\,\epsilon_{z,t} \tag{4.10a}$$

where $\epsilon_{z,t}$ is a i.i.d. shock distributed according $\Upsilon\left(0, \left(1-\gamma^2\mathbf{E}(E^{i_z}_{t-1}\pi_t - \overline{\pi}_t)^2\right)/(1-\gamma)^2\right)$ so that overall variance is simply $\mathbf{E}\left(v_z^2\right) = \delta$. The specific forms of observational errors as maintained by (4.10a) do not add nothing substantial in the general framework. The coefficient γ controls for the covariance between this observational error, and the estimated distance of the actual inflation from the estimated fundamental one. Note that the latter is a proxy for the amount of behavioural uncertainty in the economy. In other words, this specification takes account of the idea that the public receives a biased information whose idiosyncratic component is possibly further amplified or dumped in the transmission. Finally, from aggregation over Z we have

$$\int_{z\in Z} v_z\ dz = \frac{\gamma}{2}\left(\sum_{i=1,2}(E^i_{t-1}\pi_t - \overline{\pi}^e_t)\right), \tag{4.11}$$

so that the aggregate expectation $E_{t-1}\pi_t$, is now equal to

$$E_{t-1}\pi_t = \overline{\pi}^e_t + \frac{(1+\gamma)}{2}\sum_{i=1,2}\left(E^i_{t-1}\pi_t - \overline{\pi}^e_t\right), \tag{4.12}$$

that is, it depends on both institutional forecasters' expectations of displacements of actual inflation rate from the estimated fundamental one *and* on the neutrality of the information channel measured by γ.

4.3.3 The actual law of motion

From (4.9) it is simple to check that institutional forecasters' expectations can be expressed as linear function of the fundamental price and observational errors. Formally we have

$$E^1_{t-1}\pi_t = \overline{\pi}^e_t + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} + \frac{b_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} \tag{4.13a}$$

$$E^2_{t-1}\pi_t = \overline{\pi}^e_t + \frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} \tag{4.13b}$$

provided $bc \neq 1$. Processes (4.13) cannot be inferred by agents since they cannot distinguish observational errors. According to (4.2), (4.12) and (4.13) makes the actual law of motion to move according to the following process

$$\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\overline{\pi}^e_t + \frac{\beta^*}{2}\left(\frac{b_{t-1}\left(1+c_{t-1}\right)}{1-b_{t-1}c_{t-1}}v_{1,t-1} + \frac{c_{t-1}(b_{t-1}+1)}{1-b_{t-1}c_{t-1}}v_{2,t-1}\right) + \nu_t \tag{4.14}$$

where $\beta^* \equiv \beta(1+\gamma)$. Notice that $\pi_t = \boldsymbol{\alpha}'\mathbf{z}_{t-1} + \beta\overline{\pi}^e_t$ if: *i*) agents are not uncertain about others' behavior, that is $v_{1,t-1} = 0$ and $v_{2,t-1} = 0$, or *ii*) agents hold the rational expectation, that is, $b_{t-1} = 0$ and $c_{t-1} = 0$, or *iii*) expectations have a zero impact on the actual course given $\beta^* = 0$. In such cases the problem reduces to the simple one extensively studied in classical adaptive learning literature in relation to the cobweb reduced form. Otherwise inflation will exhibit endogenous excess volatility around the estimated inflation due by the stochastic term originated by non linear combination of observational errors and possibly further amplified (or dumped) by the transmission channel term impacting through β^*.

4.4 Equilibria and Learneability

4.4.1 Equilibria

Equilibria are such that institutional forecasters' forecast errors are orthogonal to available information, namely to exogenous variables time series and noisy perceptions of others' expectations. Formally they have to solve the following system

$$\begin{aligned}
\mathbf{E}[\mathbf{z}_{t-1}(\pi_t - T'_{\mathbf{a}}\mathbf{z}_{t-1})] &= 0 \quad (4.15)\\
\mathbf{E}[\left(E^2_{t-1}\pi_t + v_{1,t-1} - \overline{\pi}^e_t\right)\left(\pi_t - \overline{\pi}^e_t - T_b\left(E^2_{t-1}\pi_t + v_{1,t-1} - \overline{\pi}^e_t\right)\right)] &= 0 \quad (4.16)\\
\mathbf{E}[\left(E^1_{t-1}\pi_t + v_{2,t-1} - \overline{\pi}^e_t\right)\left(\pi_t - \overline{\pi}^e_t - T_c\left(E^1_{t-1}\pi_t + v_{2,t-1} - \overline{\pi}^e_t\right)\right)] &= 0 \quad (4.17)
\end{aligned}$$

where $T_{\cdot}$ map gives the coefficients of the linear forecast rule yielding local minima of the mean square error variance conditioned on the available information set. For a technical reference on projections and convergence properties of adaptive learning algorithms used in what follows, see Marcet and Sargent (1989), Evans and Honkapohja (2001).

Proposition 26 *T-map takes the form*

$$\begin{aligned}
T_{\mathbf{a}}(\mathbf{a}) &= \boldsymbol{\alpha} + \beta\mathbf{a}\\
T_b(b,c) &= \frac{\beta^*}{2}\left(\frac{b(1+c)(1+c\rho_v) + c(1+b)(c+\rho_v)}{1+c^2(1+2\rho_v)}\right)\\
T_c(b,c) &= \frac{\beta^*}{2}\left(\frac{b(1+c)(b+\rho_v) + c(1+b)(1+b\rho_v)}{1+b^2(1+2\rho_v)}\right)
\end{aligned}$$

Proof. Keep in mind that observational errors $v_{i,t-1}$ have zero mean and they are uncorrelated with exogenous variable $\mathbf{z}_{t-1}$, that is $E[v_{i,t-1}\mathbf{z}_{t-1}] = \mathbf{0}$. Spelling out conditions for $T_{\mathbf{a}}$ and T_b (T_c is mirror like),we have respectively

$$T_{\mathbf{a}} : \boldsymbol{\alpha}'\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] + \beta\ \mathbf{a}'\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] - T'_{\mathbf{a}}\ E[\mathbf{z}'_{t-1}\mathbf{z}_{t-1}] = 0$$

and

$$\begin{aligned}
T_b : \frac{\beta^*}{2}\left(\frac{b(1+c)}{(1-bc)^2}\delta + \frac{c^2(b+1)}{(1-bc)^2}\delta + \frac{c(b+1)+cb(1+c)}{(1-bc)^2}\delta\rho_v\right) + \\
- T_b\left(\frac{1}{(1-bc)^2}\delta + \frac{c^2}{(1-bc)^2}\delta + 2\frac{c}{(1-bc)^2}\delta\rho_v\right) = 0
\end{aligned}$$

Finally the projected T map for $\mathbf{a}$, b and c is given by

$$\begin{aligned}
T_{\mathbf{a}} &= \boldsymbol{\alpha} + \beta\mathbf{a}\\
T_b &= \frac{\beta^*}{2}\left(\frac{b(1+c)(1+c\rho_v) + c(b+1)(c+\rho_v)}{1+c^2+2c\rho_v}\right),\\
T_c &= \frac{\beta^*}{2}\left(\frac{b(1+c)(b+\rho_v) + c(b+1)(1+b\rho_v)}{1+b^2+2b\rho_v}\right).
\end{aligned}$$

■

Notice T-map depends on error variances ratio $\varepsilon_1/\varepsilon_2$ and not at all on the extent of them. Moreover if $\varepsilon_1 = \varepsilon_2$ errors variances simply disappear from equations.

Since the inflation process is endogenously determined by agents' forecasts, the T-map depends on the coefficients of the forecast rules. Therefore, fixed points of the T-map are the values for which professional forecasters do not commit systematic error given available information.

Definition 27 *Equilibria obtain as fix points of the T map for* $T_{\mathbf{a}}(\widehat{\mathbf{a}}) = \widehat{\mathbf{a}}$, $T_b\left(\widehat{b}, \widehat{c}\right) = \widehat{b}$ *and* $T_c\left(\widehat{b}, \widehat{c}\right) = \widehat{c}$.

Now it is possible to state the following.

Proposition 28 *Equilibria of the system are:*
i) a REE $(\widehat{\mathbf{a}}', \widehat{b}, \widehat{c}) = ((1-\beta)^{-1}\boldsymbol{\alpha}', 0, 0)$,
ii) an high BSE $(\widehat{\mathbf{a}}', \widehat{b}, \widehat{c}) = ((1-\beta)^{-1}\boldsymbol{\alpha}', \frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)}, \frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$,
iii) a low BSE $(\widehat{\mathbf{a}}', \widehat{b}, \widehat{c}) = ((1-\beta)^{-1}\boldsymbol{\alpha}', \frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)}, \frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$.

Proof. Equilibria are given by the system:

$$\widehat{\mathbf{a}}' = \boldsymbol{\alpha} + \beta\widehat{\mathbf{a}}' \tag{4.18a}$$

$$\widehat{b} = \frac{(\beta^*/2)\,\widehat{c}(\widehat{c}+\rho_v)}{(1-(\beta^*/2)(1-\rho_v))\widehat{c}^2 + ((2-\beta^*)\rho_v - \beta^*/2)\widehat{c} + (1-(\beta^*/2))} \tag{4.18b}$$

$$\widehat{c} = \frac{(\beta^*/2)\,\widehat{b}(\widehat{b}+\rho_v)}{(1-(\beta^*/2)(1-\rho_v))\widehat{b}^2 + ((2-\beta^*)\rho_v - \beta^*/2)\widehat{b} + (1-(\beta^*/2))} \tag{4.18c}$$

assuming $bc \neq 1$. It is easily proved by substitution that the fundamental rational expectation solution it is always a rest point of the T-map. Other non fundamental $\widehat{b}$ and $\widehat{c}$ equilibria values are in correspondence of $\widehat{b} = \widehat{c}$ and result as solutions to

$$\widehat{c}\left(\widehat{c}^2\left((1-\beta^*/2) - (\beta^*/2)\rho_v\right) - (\beta^* - (2-\beta^*)\rho_v)\widehat{c} + (1-\beta^*/2) - (\beta^*/2)\rho_v\right) = 0 \tag{4.19}$$

featuring respectively the high BSE values (b_+, c_+) and the low BSE values (b_-, c_-) where

$$c_+ = b_+ = \frac{\beta^* - (2-\beta^*)\rho_v + 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)} \tag{4.20}$$

$$c_- = b_- = \frac{\beta^* - (2-\beta^*)\rho_v - 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)} \tag{4.21}$$

exist whenever $\beta^* \geq 1$. ■

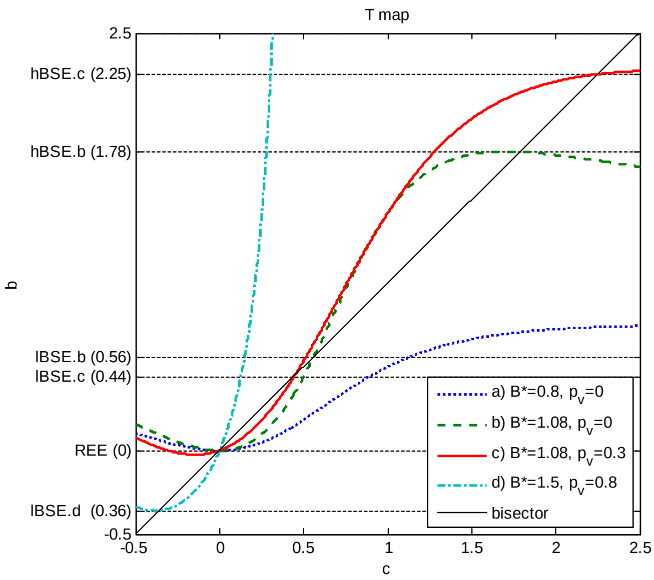

Figure 4.3: Tmap representation for different calibration. Equilibria are at intersection with Tmap with the bisector. For values of β^* bigger than one two BSEs arise besides REE.

The figure below plots T_b for four different calibrations. Given the symmetric nature of the problem we are analyzing, BSEs are at the intersection of T_b with bisector. Line a is obtained for $\beta^* = 0.8$ and $\rho_v = 0$. In such a case the unique intersection is at the REE values $\widehat{b} = \widehat{c} = 0$. As β^* goes up to one (line b), two BSE emerge such that not trivial values of $\widehat{b}$ and $\widehat{c}$ exist such that forecast errors are uncorrelated with available information. Ceteris paribus increasing values of ρ_v (line c) makes the BSE with smaller values closer to REE values and the high one being further away. Finally, extreme calibration as the one showed by line d yields negative BSEs; the low one is shown in the picture.

The arising of equilibria different from the REE is due to non-linearity of the T-map induced by the non-linear constraint linking observational errors as they appears in (4.9). In this respect, BSEs are kind of limited-informed rational expectation (Sargent 1991), because agents cannot observe all the stochastic components of the actual law of motion separately. The non-linear link between observational errors generates externalities to the individual forecasting problem. In fact, BSE are kind of coordination failures in that once achieved variance of observational errors transmits persistently to the course of actual output gap through aggregate expectation, making the overall forecast error variance higher then the REE one, even if locally minimal. Such equilibria do not require any external coordinating mechanism or common knowledge assumption. They arise as the result of endogenous coordination among non cooperative agents. In that respect they are qualitatively different from classical sunspot equilibria. For further details about BSE see Gaballo (2009).

4.4.2 Learneability

This section explores leaneability of REE and the possibility of adaptive learners being stuck in a BSE, that is whether or not BSE are learneable . The concept of learneability refers to the nature, stable or unstable of the learning dynamics around the equilibria computed above under a recursive least square algorithm .

Definition 29 *An equilibrium* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *is locally learnable under recursive least square (RLS) algorithm if and only if there exist some neighborhood* $\Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *of* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *such that for each initial condition* $(\mathbf{a}_0,b_0,c_0) \in \Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *the estimates converge almost surely to the equilibrium, that is* $\lim_{t\to\infty}(\mathbf{a}_{t-1},b_{t-1},c_{t-1}) \overset{a.s.}{=} \left(\widehat{\mathbf{a}},\widehat{b},\widehat{c}\right)$.

To check learneability one need to investigate the Jacobian of the T-map. If the matrix of all partial derivative of T-map in the equilibrium has all eigenvalues lie inside the unit circle, we can say the equilibrium to be stable under learning (Marcet and Sargent 1989, Evans Honkapohja 2001). The Jacobian for T-map takes the form

$$JT(\mathbf{a},b,c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \frac{dT_b(b,c)}{db} & \frac{dT_b(b,c)}{dc} \\ 0 & 0 & \frac{dT_b(b,c)}{dc} & \frac{dT_c(b,c)}{dc} \end{pmatrix}$$

where

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}\frac{(1+c)(1+c\rho_v)+c(c+\rho_v)}{1+c^2+2c\rho_v}, \tag{4.22a}$$

$$\begin{aligned}\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}&\frac{b(1+c\rho_v)+b\rho_v\,(1/2)\,(1+c)+(c+\rho_v)(1+b)}{1+c^2+2c\rho_v}+ \\ &-\frac{2(c+\rho_v)(b(1+c\rho_v)(1+c)+c(c+\rho_v)(1+b))}{1+c^2+2c\rho_v},\end{aligned} \tag{4.22b}$$

$$\begin{aligned}\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}&\frac{c(1+b\rho_v)+c\rho_v\,(1/2)\,(1+b)+(b+\rho_v)(1+c)}{1+b^2+2b\rho_v}+ \\ &-\frac{2(b+\rho_v)(c(1+b\rho_v)(1+b)+b(b+\rho_v)(1+c))}{1+b^2+2b\rho_v},\end{aligned} \tag{4.22c}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}\frac{(1+b)(1+b\rho_v)+b(b+\rho_v)}{1+b^2+2b\rho_v}. \tag{4.22d}$$

To analyse learneability of equilibria we have to investigate the sign of eigenvalues of the matrix $K \equiv JT - I$ (where I is the identity matrix) in the equilibrium values $\widehat{\mathbf{a}}$ and

$\widehat{c} = \widehat{b}$ given by

$$K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta - 1 & 0 & 0 \\ 0 & 0 & \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 & \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} \\ 0 & 0 & \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} & \left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} - 1 \end{pmatrix}, \tag{4.23}$$

with

$$\begin{aligned} \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 &= \frac{((\beta^*/2)\,(1+\rho_v) - 1)\,\widehat{b}^2 + ((\beta^*/2)(1+2\rho_v) - 2\rho_v))\,\widehat{b} + (\beta^*/2) - 1}{1 + \widehat{b}^2 + 2\widehat{b}\rho_v}, \\ \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= (\beta^*/2)\,\frac{(2\rho_v^2 - 1)\widehat{b}^3 + 3\rho_v^2\widehat{b} + 3\widehat{b} + \rho_v}{\left(1 + \widehat{b}^2 + 2\widehat{b}\rho_v\right)^2}, \\ \left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} \quad \text{and} \quad \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} = \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})}. \end{aligned}$$

A certain equilibrium $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ is learnable if and only if the matrix $K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})}$ has all negative eigenvalues. Figure 4 below shows numerical analysis for the whole parameter range[4] spanned by β^* and ρ_v. Keep in mind that a necessary condition for learneability of equilibria is always $\beta < 1$. We presume it in the following discussion.

As is evident from inspection of the plot REE is the only learneable equilibrium in the region $\beta^* < 1$. In the white area a learneable high BSE (hBSE) arises besides a REE. This area is the most interesting in that it partially includes most realistic calibration values for the Lucas-type monetary model. Notice that whenever a hBSE exists it is not the unique learneable equilibrium. For such values the learning mechanism selects between REE and hBSE. How this happens will be explained in detail later, when we will present numerical simulation of the dynamic system. REE and hBSE are both learneable for lower values of β^* as ρ_v increases in modulus. Specifically, as ρ_v approach unity for sufficiently high value of β^* the low BSE (lBSE) becomes learneable and both REE and hBSE are no longer. This is a standard property of non linear dynamics: given the system has three equilibria, either the most distant two are dynamically stable or only the one

[4] From quite immediate application of a proposition proved in Gaballo (2009), REE solution $(\widehat{\mathbf{a}},\widehat{b},\widehat{c}) = (0, \frac{\alpha}{1-\beta}, 0, 0)$ is learneable whenever

$$\begin{aligned} \beta &< 1, & (4.24) \\ \beta^* &\leq \frac{2}{1+\rho_v} \quad \text{with } \rho_v \geq 0, & (4.25) \\ \beta^* &\leq \frac{2}{1-\rho_v} \quad \text{with } \rho_v < 0. & (4.26) \end{aligned}$$

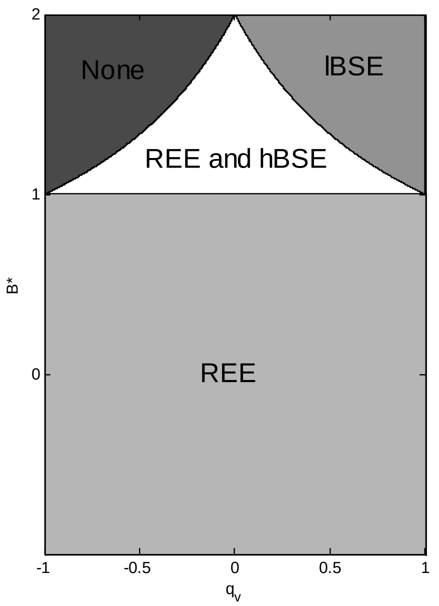

Figure 4.4: Numerical learneability analysis in the whole parameter space. The space is partitioned in four regions exhibiting different learneability properties. In the white one REE and the high BSE are both learneable and learning dynamics select among them. In the light grey only REE is learneable whereas in the dark grey only the low BSE is learneable. In the black area none learneable equilibria are present.

in the middle is dynamically stable (refer to figure 3). On the other hand as ρ_v decreases for sufficiently high value of β^* the system presents no learneable equilibria.

Whenever learneable BSEs exist, distances between equilibria measured on the bisector line in figure 3 are indicative of the size of basins of attraction. In particular, at least in the range considered, the T-map behaves like a cubic yielding three dynamic equilibria. As usual, the middle equilibrium either is unstable and works as threshold between the basins of attraction of the other (stable) two or is the unique stable equilibria with a basin of attraction lying between equilibria at the extremes. As example for calibration b and c in figure 3, the REE and the high BSE basins of attraction are divided by the low BSE. In particular as β^* increases REE basin of attraction shrinks, whereas the high BSE one enhances.

Finally, notice that necessary condition for arising and learneability of BSEs in this model is that $\gamma > 0$, that is, private sector agents commits observational errors that are correlated in average with the idiosyncratic departures of actual price from the fundamental one forecasted by institutional forecasters. In that respect transmission of information in the economy plays an essential role. What matters is that forecasted idiosyncratic departure from the fundamental rate of inflation are amplified by private sector overreaction. Other behavioral schemes of the private sector can be actually implemented to obtain the same, or more complex, dynamics.

4.4.3 Excess volatility

Equilibria with $\left(\widehat{b},\widehat{c}\right) \neq (0,0)$ includes extra stochastic variables, namely observational errors, with no economic content in agents' expectation function. BSE present, as any sunspot solution, a volatility higher than the fundamental solution. The extent of theoretical excess variance is measured in equilibrium $\left(\widehat{b}=\widehat{c}\right)$ by

$$2\left(\frac{\beta^*\widehat{b}}{2\left(1-\widehat{b}\right)}\right)^2(\varrho+\rho_v),$$

and it is increasing in β, δ and ρ_v and decreasing in $\widehat{b}$ for $\widehat{b}>1$.

The picture below plots excess variance yield by learneable BSE in terms of observational error variance for values $\beta^* \in (1,2)$ ("5" stays for "5 and more"). For values close to unity excess variance is really high but it decreases very soon. The most part of the relevant region exhibits an excess volatility between one and four times the variance of observational errors. In the region for which REE and the high BSE are both learneable we can have different volatility regimes (an high volatility one being in correspondence of the high BSE) depending on the equilibrium selected by the learning algorithm. Next section we finally explain and show how unpredictable and endogenous switching of volatility regimes can be triggered by constant gain algorithm.

4.5 Constant gain learning simulation

The simulations proposed in this section provides examples of endogenous and unpredictable changes in volatility regimes. We chose calibrations such that analytical results can be contrasted with experiments. All simulations are generated with the following parameter setting: $\beta=0.8$, $\delta=\varphi_2$, $\delta=0.1$. The exogenous shocks are all Gaussian white noises with unit variance. In all figures the following conventions hold. In the upper box is displayed the dynamics of the two coefficients, b_t and c_t. Whenever low BSE values serve as divide between REE and high BSE basins of attraction, these are indicated by a dotted line in the upper box. The lower box shows the corresponding dynamics of both actual inflation π_t and agents' estimated fundamental inflation (flatter line). The first four figures are generated with the same series of errors and with same initial conditions closely set around REE value.

Figure 6 displays the benchmark case, that is, convergence in distribution to REE

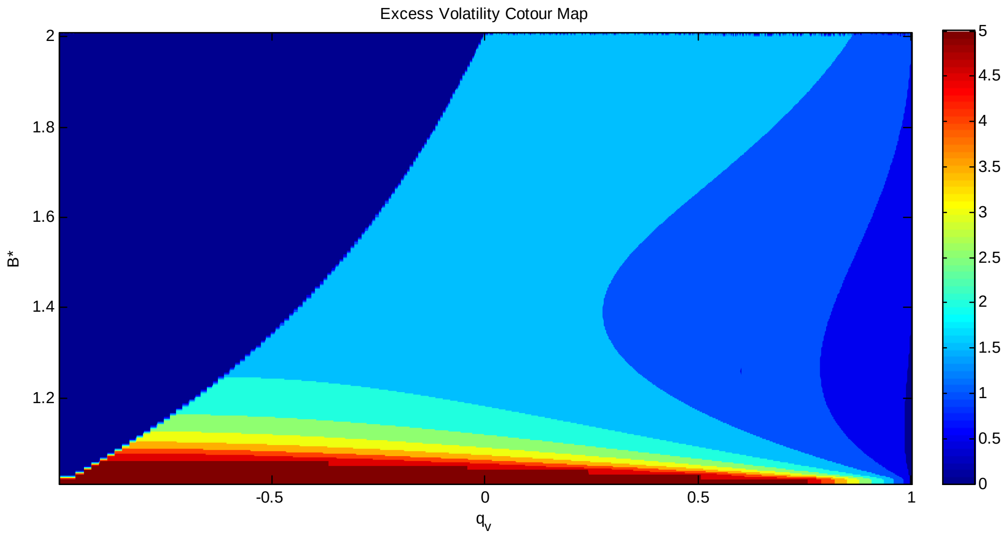

Figure 4.5: Numerical analysis of excess volatility. The picture shows the size of excess volatility obtained for values $\beta^* \in (1, 2)$ for which learneable BSEs arise. The unit of measure of the scale is the variance of observational errors. ("5" stays for "5 and more").

values for $\rho_v = \gamma = 0$. The gain is settled $g_d = g_f = \delta^{-1}/110$. The factor δ^{-1} has been included in the gain so that the adjustments of both b_t and c_t are substantially equal to the ones obtained with constant gain OLS around REE values for $\overline{g} = 110$.. Notice how constant gain learning generate continuous small displacements away from REE values. Nevertheless such diplacements are temporary escapes and do not substantially affect the variance of actual inflation process. In the second box one can appreciate the near-natural REE variance and how the estimate of fundamental inflation soon approaches the REE value.

In figure 6 the calibration of figure 1 is modified only in that $\gamma = 0.28$ (so that $\beta^* = 1.08$). For such values one learneable high BSE arise for $b = c = 1.78$. Up to 1300 periods the dynamics is roughly the same, but how estimates approach low BSE values, the dynamics changes dramatically. In particular as estimates overcome low BSE values (around 2000 periods) the dynamics enters in the basin of attraction of the high BSE making estimates to converge in distribution to it. This endogenous structural change affect in a persistent and substantial way actual inflation variance. The resulting excess variance is about three times REE variance. Notice how excess volatility generated by high BSE affects volatility of the estimated fundamental inflation too, contributing to the overall variance of actual inflation. This effect is more evident in next and last pictures.

Figure 8 is generated with same setting introducing a small correlation between ob-

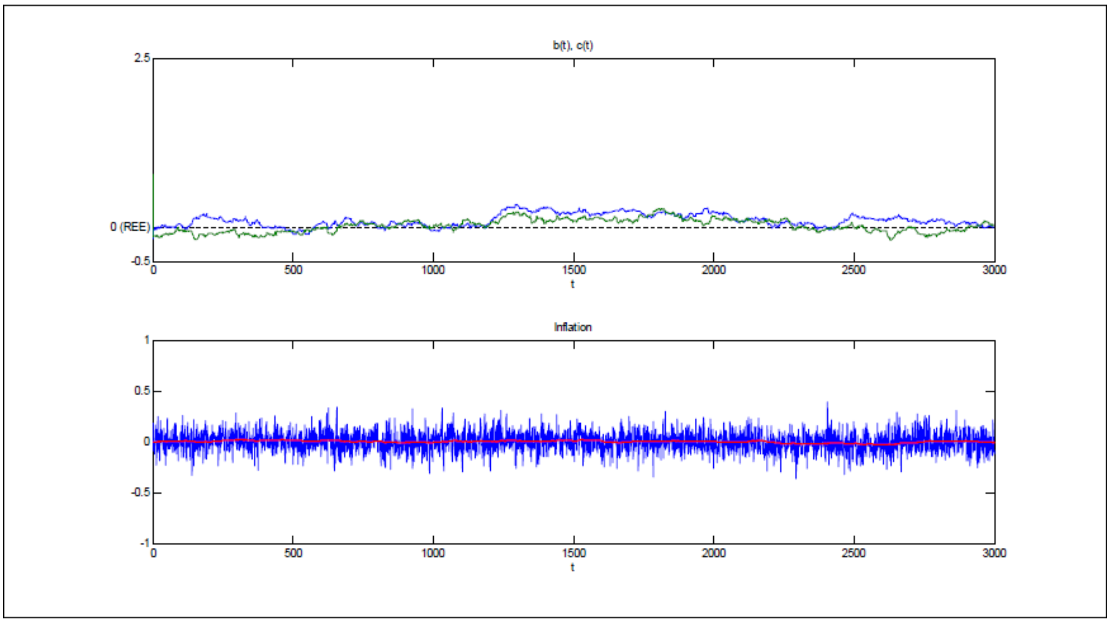

Figure 4.6: Benchmark case. Convergence to REE ($\beta = 0.8$, $\gamma = 0$, $\rho_v = 0$). Line a). in figure 3.

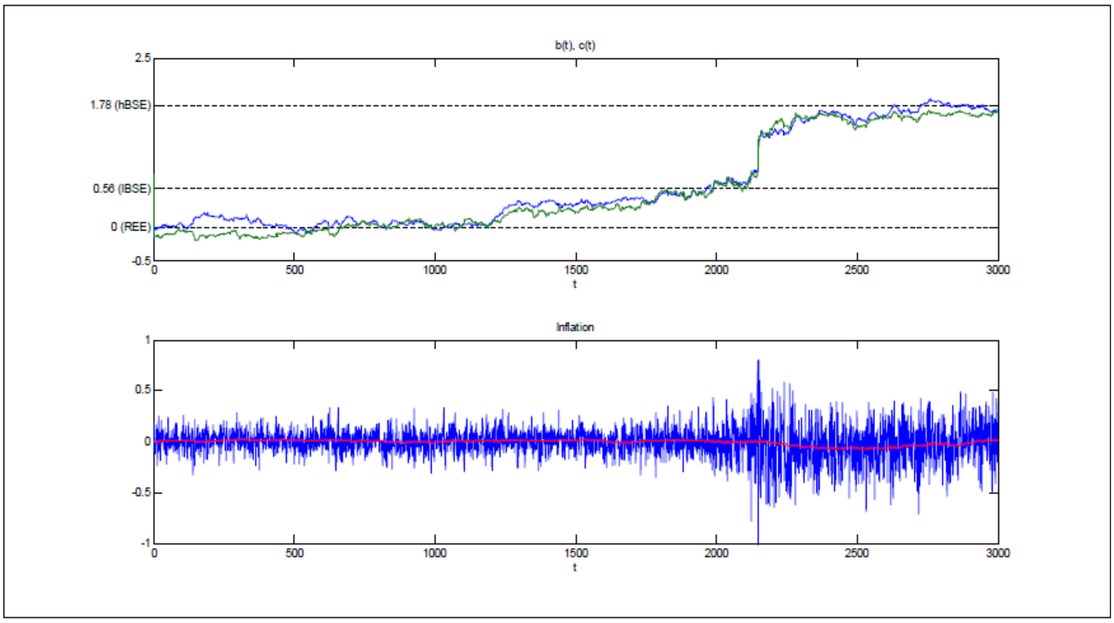

Figure 4.7: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0$). Line b). in figure 3.

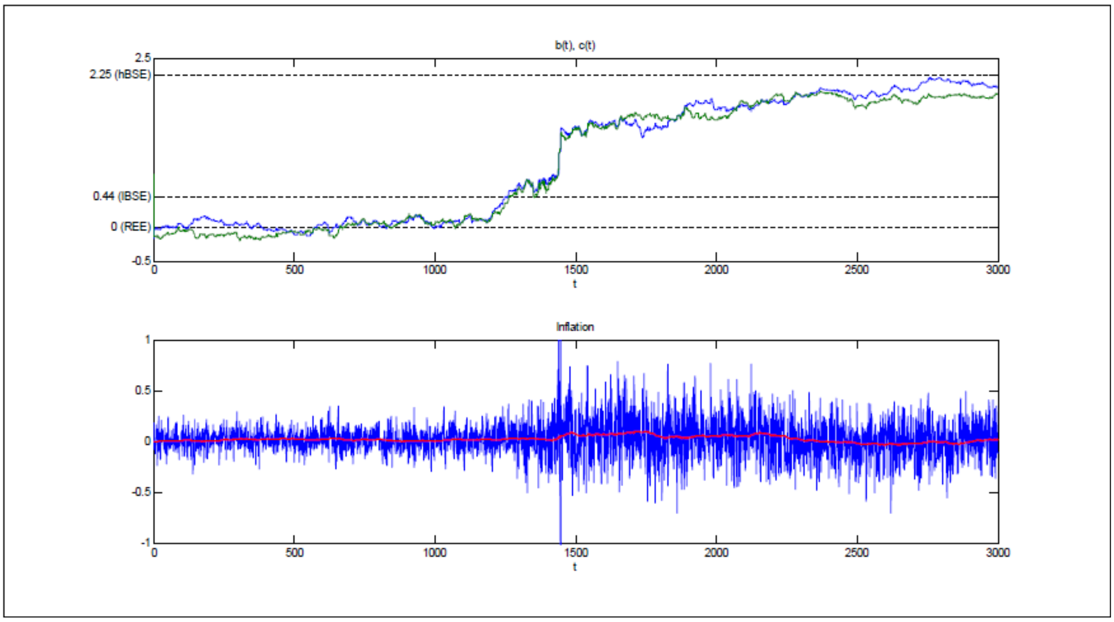

Figure 4.8: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0.3$). Line c). in figure 3.

servational errors $\rho_v = 0.3$. The effect of this type of correlation is in a earlier jump to the correspondent learneable high BSE. This is not surprising since for increasing positive values of ρ_v the corresponding high BSE values increase and low BSE ones decrease. This means that high BSE basin of attraction enhances and REE basin shrinks, so that jumps from REE to high BSE is more likely to happen. As already noted, it is possible to appreciate this feature contrasting line b and c in figure 3.

Figure 9 provides an example of convergence to the low BSE. This occurs for quite extreme and careful calibration in that low BSE basin of attraction is quite narrow given the closeness of low BSE values to REE ones. The one displayed is obtained for $\gamma = 0.7$ and $\rho_v = 0.8$. As evident the contribute to overall actual inflation variance is almost negligible. Finally last picture shows how with appropriate calibration is it possible to obtain a series of endogenous and unpredictable switches from REE to high BSE and viceversa. Several features contribute to the aim. Firstly correlation coefficient are $\gamma = 0.21$ (that makes $\beta^* = 1.01$ very near unity) and $\rho_v = 0.4$. For such values low BSE values are about half of high BSE ones, that in turn result to be relatively quite small. Therefore REE and high BSE basins of attraction have almost the same extent. Moreover we chose a bigger gain, namely $g_d = g_f = \varrho^{-1}/48$ in order to make estimates dynamics more volatile and hence jumps more likely.

Numerical simulation shows that dynamics similar to the latter can be generated considering more than two institutional forecasters with less extreme calibration[5]. Ana-

[5]The programm is available upon request to the author.

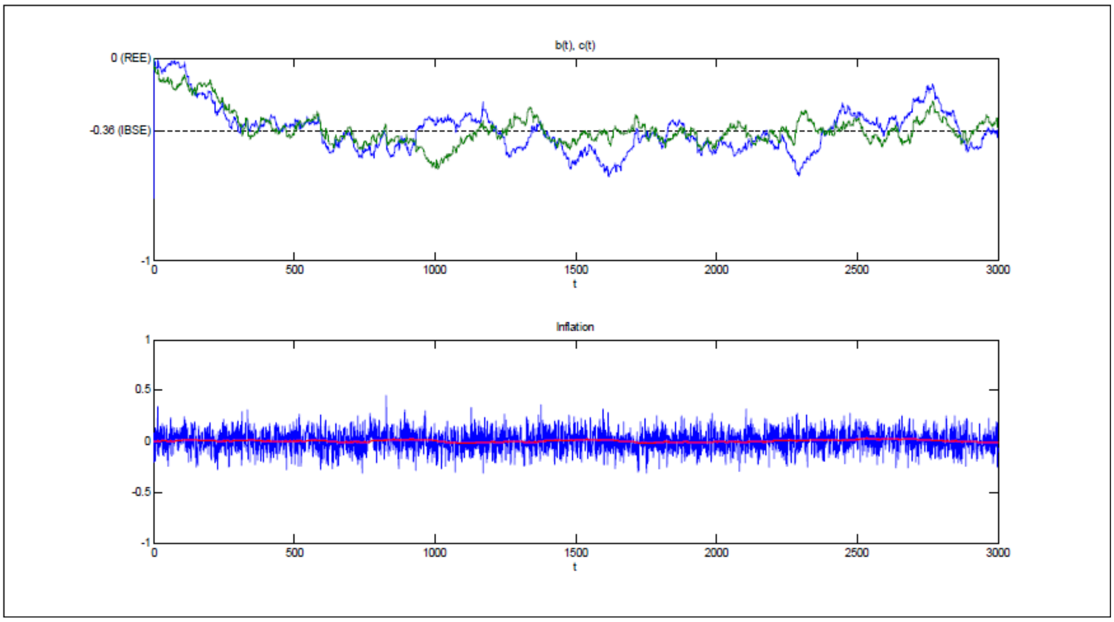

Figure 4.9: Convergence to lBSE ($\beta = 0.8$, $\gamma = 0.7$, $\rho_v = 0.8$). Line d). in figure 3.

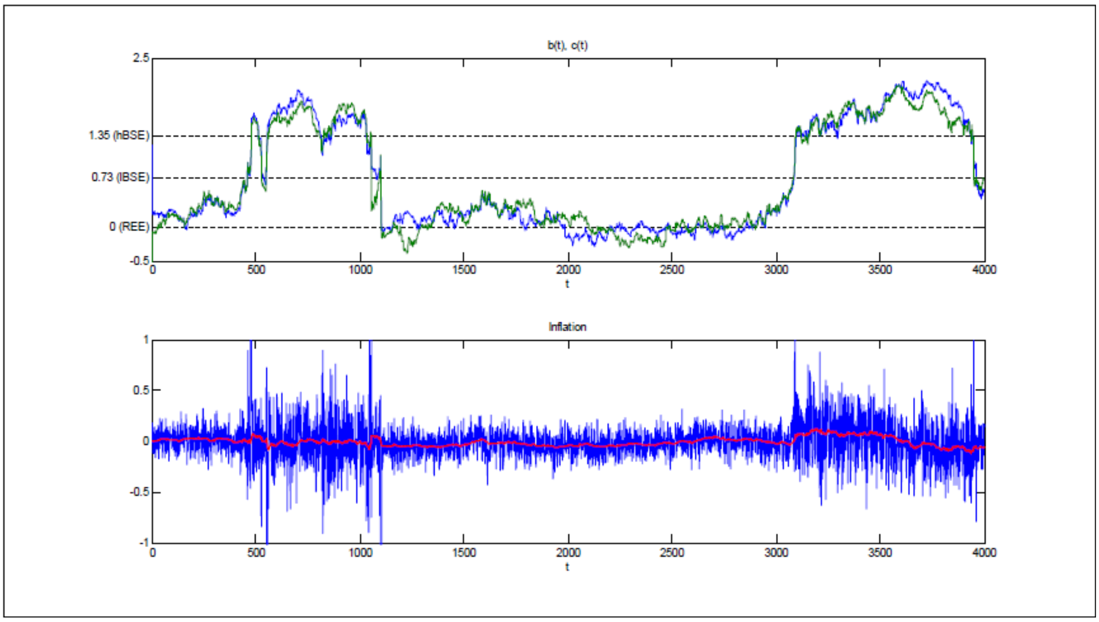

Figure 4.10: From REE to hBSE and back two times ($\beta = 0.8$, $\gamma = 0.21$, $\rho_v = 0.4$).

lytical results for such cases require a quite cumbersome computational analysis that is far beyond the scope of this work and will be object of future investigation.

4.6 Conclusion

Adaptive learning in macroeconomics has been always presented as a bounded rationality approach since a central hypothesis is that agents don't recognize the self-referential nature of the model. In other words, agents focus only on exogenous determinants of the economy by-passing all issues linked to interactions among them. This feature results as an ad-hoc departure from full rationality paradigm and, as such, it weakens the theoretic robustness of this approach. More importantly the bounded rationality hypothesis prevents the explicit modelling of interdependence between agents' expectations that is widely recognized to be responsible for crisis triggering. Gaballo (2009) shows how to extend the approach to deal with such issues. Here we have used such results to model endogenous changes in volatility regimes due to emergence of interdependence among agents' expectations. We have also shown a simple way to reconcile the standard use of adaptive learning approach with the idea agents recognize the self-referential nature of the economy.

We have investigated a simple Lucas-type monetary model in which inflation depends on expectation of current inflation and other exogenous determinants. In this setting we assumed expectations are interdependent in two respects. Firstly private sector evenly relies on two institutional forecasters. The latter are the only ones among agents having resource to gather and efficiently analyse information. In fact, each institutional forecaster implements statistical techniques to learn in real time the rational expectation, that is the fundamental inflation. The second way by which expectations are interdependent is due to behavioral uncertainty hypothesis. Behavioral uncertainty means that each institutional forecaster does not have perfect information about the other one's simultaneous expectations, but only a noisy signal of it. Given non-negligibility of institutional forecasters' expectations, they have incentive to condition their expectations to these noisy signals in order to minimize their forecast error variance. In particular, they have to assess whether or not actual deviations from the esteemed fundamental rate are due to idiosyncratic departure of others rationality from the rational one. In sum, institutional forecasters have to learn not only about the fundamental inflation rate (as in standard adaptive learning literature) but also about rationality of others.

We have proved how the interaction of these two channels of expectations interdependence and constant gain adaptive learning can give rise to two type of learneable equilibria, namely the REE and BSEs. The former occurs whenever both agents' estimates of the optimal weight of noisy behavioral observations converge in distribution to zero, the latter arises otherwise. BSE are equilibria for which volatility of behavioral noisy observations enters in the actual law of motion generating excess inflation volatility. More importantly, constant gain learning generates endogenous, unpredictable and persistent switches in volatility regimes. These changes are obtained without any aggregate shock exogenously imposed. On the contrary excess volatility is triggered by noises justified by behavioral uncertainty at a micro level. The model has the advantage of being perfectly consistent with REE behavior and, nevertheless, it has the potentiality to exhibit endogenous structural changes.

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