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ABSTRACT

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The purview of this dissertation is the positive and the normative analyses of general disequilibrium. Our spur for focusing on the theory of disequilibrium is the persistent empirical non-clearing market phenomenon. In particular, disequilibrium in the form of significant unemployment and underutilized capacities is omnipresent in all market economies. Yet the Walrasian equilibrium - the predominant equilibrium theory of markets – is inconsistent with non-clearing markets by definition.

Motivated by these considerations, we endeavor to elucidate theoretically why markets do not clear, and scrutinize the efficiency implications of disequilibrium. To this end, we focus solely on convex economies with continuous preferences and production technologies – the standard domain of the Walrasian analysis. We also posit the Walrasian behavioral proviso according to which all agents are price-taking optimizers.

Our first theoretical contribution is to demonstrate that the Walrasian equilibrium does not exist under certain hypotheses. That is, conditional on our assumptions, market clearing is an impossibility which can potentially explain persistent real world disequilibrium. As far as we are concerned, this is the first study that gives a set of general conditions inducing non-existence of Walrasian equilibrium.

Although the non-existence of Walrasian equilibrium can explain why markets do not clear, it is not competent to analyze how resources would be allocated in disequilibrium. This observation leads us to the inquiry of the non-Walrasian equilibrium – a generalization of the Walrasian equilibrium engineered to analyze non-clearing markets. Our second contribution is to prove that the non-Walrasian equilibrium exists albeit the Walrasian counter-part may not. To the best of our knowledge, this result also gives the most general existence conditions for the non-Walrasian equilibrium concept.

Finally, we move to the inquiry of the efficiency properties of non-clearing markets. That disequilibrium is inefficient is a common verdict among economists. Indeed, we demarcate the conditions that ensure disequilibrium allocations are Pareto-inefficient. All known theorems in this field (i.e. efficiency analysis of disequilibrium) are special cases of our study.

These results admit the following claim to hold. There are economies where market clearing is impossible which induces disequilibrium and ensures Pareto-inefficiency if there is no intervention to the market.

An intriguing policy implication ensues by applying the Second Fundamental Theorem of Welfare (SFTW) which asserts that any Pareto-efficient allocation can be supported as a Walrasian equilibrium with a proper lump-sum redistribution of income. Thus, in virtue of the SFTW, there is always a Pareto-improving redistribution scheme for the economies that satisfy our assumptions.

We believe these theoretical discussions are germane to the ancient polemics of markets' competency of self-regulation to maintain equilibrium, and the efficiency of redistributive policies. Our results show that even unfettered markets with the most flexible adjustment mechanism may not guarantee equilibrium and redistributive policies may very well be efficiency enhancing – an argument almost vertically opposing the mainstream view.

The dissertation is comprised of the following chapters. The first chapter is a survey of the literatures to which we make theoretical contributions. The second chapter is about the non-existence of Walrasian equilibrium. The third chapter deals with the existence of non-Walrasian equilibrium. The forth chapter is about the efficiency properties of disequilibrium. Finally, we conclude with a brief discussion of the subjects covered in above chapters mainly underscoring the weaknesses of the results involved.

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CHAPTER 1

A Survey of Non-existence of Walrasian Equilibrium, Existence of Non-Walrasian Equilibrium, and Disequilibrium Welfare Analysis

1.1. Introduction

Perfectly competitive equilibrium *à la* Arrow and Debreu, which is also known as Walrasian equilibrium, is widely deemed the formalization of how unfettered markets operate on competitive principles. Needless to say, it is impossible to exaggerate the significance of such a theoretical framework if it is consonant with observed data.

However, the theory of competitive equilibrium suffers significant and persistent empirical failure in its predictions. For disequilibrium (i.e. non-clearing market phenomenon) in the form of unemployment and underutilized capacities is palpable and ubiquitous (see Arrow (2005, p.15) and Drèze (1997)). Nor do the conditions of the existence of perfectly competitive equilibrium enjoy a strong support from data. In particular, the assumptions in the existence literature regarding the distribution of initial resources are quite heroic as we shall discuss with further details in the sequel. Hence, given a realistic distribution of initial resources, competitive equilibrium may not exist.

If competitive equilibrium does not exist then markets would not clear at any price – an implication that can explain why actual markets do not clear. The primary task of this chapter is to survey the literature of non-existence of competitive equilibrium motivated by its explanatory potential.

Then we review the literature of non-Walrasian equilibrium which is a generalization of competitive equilibrium that is engineered to analyze non-clearing markets. The aim of this addendum is to discuss whether the current state of non-Walrasian literature is competent to analyze disequilibrium that appears due to non-existence of Walrasian equilibrium.

Finally, we present a survey of disequilibrium welfare analysis which scrutinizes the efficiency implications of non-clearing markets. Therefore, established results in this literature can be used to conduct efficiency analysis of disequilibrium due to competitive equilibrium's non-existence.

It is noteworthy that this survey is not confined to bibliographical references of these three literatures albeit we endeavour to give an account of the relevant references as complete as possible. In fact, the bulk of this chapter is devoted to critical evaluations of these three literatures. More specifically, we take pains to state our methodological standpoint in equilibrium theory and discussions of existence whereby we critically evaluate the standard assumptions of the aforementioned literatures.

By way of prologue to our discussion, a general methodological discussion on equilibrium's existence ensues as promised.

1.2. Existence and non-existence of equilibrium

In this section, we define the following terms that will be frequently used in this chapter: equilibrium, existence of equilibrium, and subsequently, non-existence of equilibrium. Two caveats are in order in this regard. Firts of all, these definitions apply to equilibrium modeling in general, and is not restricted to any form of equilibrium notion – particularly competitive equilibrium- in any sense. Second, we concede that these definitions are dependent on our methodological view, and thereby, do not represent any form absolute truth.

We say that an equilibrium concept is a mapping that relates the exogenous to the endogenous. The endogenous is the predicted behavior of the decision makers who are designated by the exogenous. Thus an equilibrium concept theoretically relates the exogenous data of decision makers to their endogenous behavior.

However, analysis predicated on equilibrium behavior can follow only if the equilibrium concept is well-defined. That is, the existence of equilibrium behavior is the *sine qua non* for any scrutiny based upon equilibrium. This observation motivates proving existence of equilibrium.

Proving existence of equilibrium is demarcating the set of exogenous which is consistent with the equilibrium concept. In a similar vein, non-existence of equilibrium is an inconsistency between the exogenous data and the economic principles that constitute the equilibrium concept.

Our interest in non-existence of equilibrium stems from its following intriguing predictive power. Consider an environment corresponding to a given exogenous data, and an arbitrarily chosen equilibrium concept. Now assume that the observed choices of decision makers in this environment systematically contradict the equilibrium concept. Non-existence of equilibrium can potentially explain such a phenomenon - contradiction between observed choices and postulated theory. In other words, observed choices may contradict a particular equilibrium concept for the equilibrium concept is inherently inconsistent with the environment.

Even though this subject could be deepened far further, the current state of our discussion is sufficient for our purposes. Now we turn to applying the general arguments in this section to the specific case of perfectly competitive equilibrium and observed persistent disequilibrium in real life.

1.3. Competitive equilibrium

In the Walrasian tradition, the set of exogenous is referred to as an "economy." An economy is comprised of consumers and firms. A consumer is a tuple of initial endowments, ownership shares over firms, and preferences over products. A firm is a production technology that relates inputs into outputs.

The Walrasian equilibrium is the utility maximizing consumption plans of consumers, profit maximizing production plans of firms, and market clearing commodity prices that are taken as given by the consumers and the firms. Put it shortly, the Walrasian principles consist of optimization, market clearing, and price taking behavior. These principles are widely interpreted as rationality, price flexibility, and competitive behavior respectively.

According to the current state of the literature, these Walrasian principles are consistent with economies that are convex, closed and irreducible. That is, if an economy is convex, closed, and irreducible then there is a Walrasian (perfectly competitive) equilibrium in this economy.

Convexity and closedness are stipulated from both preference relations and production sets. Roughly speaking, convexity ensures that mixing is more utility enhancing in consumption and more efficient in production. Closedness is a topological characterization of continuity imposed on preferences and production technology.

However, our main focus is the condition of irreducibility which is germane to the distribution of initial resources among individuals, and we will focus on non-existence of Walrasian equilibrium only due to violation of irreducibility condition.

1.3.1. Irreducibility

In this section we explain the irreducibility condition which is a standard assumption to ensure existence of competitive equilibrium. There are several different irreducibility conditions in the literature (see Moore (2006), Florig (2001b), Arrow and Hahn (1971), Debreu (1962), and McKenzie (1959)). Typically these conditions are very complicated in technical terms. However, Geanakoplos (2004, p.118) provides an excellent and succinct interpretation covering all irreducibility conditions: An economy is irreducible when "each agent's labor power could be used to make another agent better off." We take Geanakoplos's interpretation as our reference point for the definition of irreducibility.

Geanakoplos's condition should hold at all feasible allocations. This implies an economy cannot be irreducible if full capacity production – the maximum output with respect to labor given non-labor resources- is feasible. That is, no known existence theorem for competitive equilibrium applies to economies with feasible full capacity production.

Let us see how feasibility of full capacity production violates irreducibility and impedes applying known existence theorems of Walrasian equilibrium. Consider an economy where full capacity is feasible, and production takes place at this level. In this case further production by increasing employment is not feasible by the definition of full capacity production. Thereby, no agent's labor power can be used to improve the well-being of another consumer.

In the light of the above, our conclusion is that known existence theorems apply only to economies where there is a shortage of labor relative to available jobs so that full capacity production is not feasible. Nevertheless, labor shortage is a very stringent condition both empirically and theoretically. This observation substantiates our interest in the possibility of non-existence of perfectly competitive equilibrium due to violation of irreducibility.

Finally, note that when competitive equilibrium does not exist because the economy is not irreducible, the problem is the distribution of initial resources. For the economy violates irreducibility only if some agents do not have enough resources valuable for others.

1.4. Non-existence of competitive equilibrium

Now we will reflect upon the studies that discuss non-existence of equilibrium due to violation of irreducibility which is pertinent to the debates of theorization and desirability of competitive markets.

The genesis of theorizing how markets operate according to the tenets of perfect competition, and of the debates on the desirability of unfettered markets have ancient roots. Adam Smith invokes the famous invisible hand parable to argue, in modern parlance, that markets induce a social optimum in virtue of decentralized optimizing behavior of consumers and firms. On the other hand, Ricardo and Say strive to provide a theoretical framework demonstrating the viability of competitive market mechanism. Their attempts laid the foundations of the modern theories of existence and stability.

Despite the elegant and systematic approach of the classicals, their view cannot account for the significant and persistent disequilibrium in real markets. This contradiction between the elegant theory and the real world phenomenon ignited the deeply influential debate of Ricardo and Say against the proto-Keynesians, to wit, Malthus and Sismondi.

According to Chipman (1965), the non-existence of competitive equilibrium in modern economic theory evokes Malthus and Sismondi's position against Ricardo and Say. Malthus and Sismondi are known to be prominent underconsumptionists. The general idea of underconsumption is that "the distributional inequalities of capitalist relations of production are inconsistent with system-wide requirements for the growth of demand and the realization of the product (Foley (1986, p.146))".

Now let us explicate Chipman's proposed connection between the underconsumptionist proto-Keynesians and the non-existence of competitive equilibrium. First of all, the state of underconsumption and the non-existence of competitive equilibrium due to violation of irreducibility stem from the same root: the original distribution of initial resources. Second, both problems point out an inherent inconsistency of the market system. The role of inconsistency for the theories of underconsumption is nicely described by Foley's definition (see above). On the other hand, non-existence of competitive equilibrium is the inconsistency between the tenets of perfect competition whereby no solution to the system of equations that represent these principles exist.

Moreover, underconsumption and non-existence of competitive equilibrium share a common policy implication as well. In particular, redistribution of income among rich and poor offers a remedy for economies that suffer underconsumption for the problem is the original distributional inequalities by assumption. In a similar fashion, redistribution of income always solves the non-existence of competitive equilibrium by the Second Fundamental Theorem of Welfare (SFTW) in the following sense. SFTW states that any Pareto-efficient outcome can be supported as a competitive equilibrium with an appropriate lump-sum redistribution scheme. Thus, even if there is no competitive equilibrium with the original distribution of resources yet there is equilibrium with redistribution.

Not only is the non-existence of competitive equilibrium pertinent to the pre-Keynesians but also to the interpretation of Keynes after the publication of his magnum opus General Theory. For example, Chipman (1965) also maintains that the non-existence of equilibrium can be deemed a violation of Say's Law - central theme in Keynesian literature. Moreover, Patinkin (1948, p.546) and Klein (1947) propose non-existence of equilibrium as a Keynesian framework to demonstrate that disequilibrium may prevail with flexible prices. Note that markets cannot restore equilibrium even with the most flexible price adjustment process if equilibrium does not exist. However, Patinkin and Klein's approach is fundamentally restricted to diagrammatic analysis with ad-hoc supply and demand functions.

To the best of our knowledge, the first rigorous example for the non-existence of competitive equilibrium with proper microeconomic foundations is due to Arrow (1952, p. 527). The exposition of the example is very brief, purely theoretical, and essentially restricted to an exchange economy.

Nonetheless, the next example in the literature makes an open reference to the economic content of the problem. In particular, Arrow and Debreu (1954) provide an example with production where establishing competitive equilibrium is impossible albeit preferences and production technology are plausible (i.e. convex and continuous). Consider an economy where production takes place according fixed coefficient technology which takes only labor and capital as inputs. Capital is given by the existing stock of initial endowments. Thus labor demand is bounded by the amount of existing capital stock. Now assume that labor supply of workers is inelastic with respect to real wages. Hence, aggregate labor supply is arbitrarily large in the number of workers. So no positive real wage would clear the labor market if the number of workers is high enough. Therefore, real wages should be zero which admit excess supply of labor in equilibrium. But no one would work when wages are zero while labor demand would be strictly positive. This shows competitive equilibrium does not exist if the numer of workers is high enough.

In this example, irreducibility condition does not hold for there are more workers then capital can employ at the full capacity production. So the labor power of some workers cannot be used to improve the well-being of some other consumers when production takes place at the full capacity. Thus, irreducibility is violated even though the prefecerences and the production technology are very standard.

The main objective of Arrow and Debreu's example is to explain that convexity and continuity are not sufficient for the existence of equilibrium, and some sort of irreducibility condition is required. However, they also remark that "[t]he possibility of disequilibrium and therefore unemployment through failure of Assumption VII [i.e. Arrow and Debreu's irreducibility assumption] to hold corresponds to so-called 'structural unemployment' (fn. 12, p. 281)". That is, Arrow and Debreu openly relate non-existence of equilibrium to disequilibrium and unemployment, in particular to structural unemployment.

Negishi (1987) provides a historical review of competitive equilibrium's non-existence. The central message of this study is that examples of competitive equilibrium's non-existence prior to Arrow (1952) are not real threats to competitive theory. An example in this line is Thornton's examples of non-existence of market equilibrium due to non-convexities. However, the non-existence example in Arrow (1952) bestows rigorous attention according to Negishi.

According to the exegesis of Rizvi (1994), irreducibility conditions unravel the fact that the generality of competitive equilibrium analysis is much-vaunted. Rizvi rightfully points out that irreducibility stipulates a particular form of interrelation between preferences, technology, and the distribution of resources. Based on this observation, Rizvi concludes that employing an irreducibility assumption causes the economy under consideration to be "held by together by non-market forces before prices can even come into play, thus limiting the coordinating function usually attributed solely to prices; and calls into question the generality of [general equilibrium] theory. (p. 18)"

The most recent discussion of non-existence can be found in Bryant (1997, 2010). Bryant offers an extensive review of all irreducibility conditions, and speculates about the possibility of violating these conditions. The message that Bryant conveys is clear and strong: The non-existence of competitive equilibrium due to violation of irreducibility is very probable and it is germane to the debate of markets' competency between the Keynesians and New-classicals. More specifically, the non-existence of competitive equilibrium is competent to explain disequilibrium phenomena.

However, there is an evident paucity of theoretical analysis in non-existence of competitive equilibrium. Many eminent theoretical economists acknowledge the possibility of equilibrium's non-existence. For example, Chichilnisky (1995) states that "[t]he problem of nonexistence of a competitive equilibrium is pervasive. Despite the fact that market allocations are regarded as a practical solution to the resource allocation problem, many standard economies do not have a competitive equilibrium." Indeed, Chichilnisky (1995) provides a general set of conditions for the nonexistence of equilibrium for exchange economies. That being said, Chichilnisky posits strongly monotonic preferences which excludes satiation any good.

But, alas, there is no study that analyzes this possibility for production economies. That is, the conditions that guarantee competitive equilibrium's non-existence are not known to the economic theory when production is involved. We believe this is a significant gap in microeconomic theory that deserves more attention due to its potential in explaining non-clearing markets.

1.5. Non-Walrasian equilibrium

Even though the non-existence of competitive equilibrium can explain why markets do not clear, it cannot be invoked to predict how resources would be allocated in disequilibrium. Non-Walrasian techniques, which subsume the Walrasian equilibrium by admitting trade out of market clearing equilibrium with arbitrarily given prices, constitute a natural candidate to approach this issue for economies without any competitive equilibrium.

First we review the literature of the non-Walrasian equilibrium concept in this section and then reflect upon its competency for the analysis of economies without any Walrasian equilibrium.

1.5.1. Preliminaries of non-Walrasian equilibrium

Non-Walrasian equilibrium concept takes preferences, technology, distribution of initial resources, and prices as the data of the economy. To make a contrast, in the Walrasian case, prices do not belong to the exogenous data but are determined through the principle of market clearing.

The exogenous data enlisted above is mapped into optimal consumption and production plans that consistently fit into each other by the non-Walrasian equilibrium. This mapping weakens the principle of market clearing in Walrasian analysis by replacing it with voluntary trade and frictionless markets while keeping optimization and price-taking behavior. More specifically, non-Walrasian equilibrium is a voluntary allocation without frictions. Now we turn to the definitions of voluntary trade and frictionless markets principles.

An allocation is voluntary if trading less is detrimental for all individuals given the budget constraints. Put it briefly, voluntariness stipulates no forced trade given the prices. For instance, in a voluntary allocation, a consumer would be worse-off if a she buys less. For the case of a worker, voluntariness prescribes that supplying less labor cannot improve well-being.[1]

On the other hand, according to the frictionless markets principle, agents on opposite sides of a particular market willing to further their trade - but failing to do so – do not exist. That is to say, any mutually beneficial trade takes place.

These two principles are natural conditions to stipulate from any market mechanism. Indeed, all major imperfectly competitive equilibrium definitions satisfy voluntariness and absence of frictions. That is to say, non-Walrasian and imperfectly competitive equilibria are isomorphic in some sense (see Madden and

[1]Therefore, that an underemployed worker would be better off if she could work more does not contradict with the definition of voluntariness.

Silvestre (1992)). Furthermore, that voluntariness is germane to how markets operate is a common view, and consequently, voluntariness is often deemed a defining tenet of market games in economic theory (see Beviá et al (2003), Svensson (1991), Benassy (1986)).

Motivated by Madden and Silvestre's analytical result, and the convention in economic theory, we argue that the set of voluntary allocations without frictions (i.e. non-Walrasian allocations) constitute an intuitive and plausible domain that cover all possible market allocations. If this view is held then non-Walrasian allocations necessarily cover all possible behavior in markets – including out of competitive equilibrium behavior- independent of the details that specify the market system.

As a result, we conclude that the definition of non-Walrasian equilibrium is admissible to analyze economies without competitive equilibrium. However, the eligibility of definition is not sufficient, and proving existence of non-Walrasian equilibrium under conditions that there is no Walrasian equilibrium is also necessary. Now we turn to this subject.

1.5.2. Existence of non-Walrasian equilibrium

As we stated above, non-Walrasian equilibrium is a voluntary allocation without frictions. However, there are several different versions of non-Walrasian equilibrium since certain subtleties arise when the technicalities are considered. These are Drèze (1975), Benassy (1975,1976), and Malinvaud and Younès (1977). Nonetheless, Silvestre (1982, 1983) proves that if Drèze equilibrium exists then other types of non-Walrasian equilibria also exist. That is, Drèze equilibrium is the strictest non-Walrasian concept. On top of that, the most widely analyzed non-Walrasian equilibrium notion is Drèze equilibrium. So we restrict our attention to Drèze equilibrium.

Drèze (1975) considers a compact cube for the set of admissible prices which is given exogenously. Thus, there are upper and lower bounds on each commodity's price which are interpreted as rigid prices. Due to these restrictions on prices, typically markets do not clear. According to Drèze's definition of non-Walrasian equilibrium, quantity constraints imposed on decision makings of individuals and firms maintain equality of consumption and production. These quantity constraints are known as rationings, and satisfy voluntariness and absence of frictions. However, this equilibrium definition does not tell which side of the market would be rationed.

In this regard, an important refinement of Drèze equilibrium is made by van der Laan (1980) and Kurz (1982) by stipulating that only supply can be rationed. This restriction on Drèze equilibrium is motivated by the fact market economies generally suffer over-supply in markets (e.g. unemployment) and over-demand is rarely observed (e.g. war times, famines). Van der Laan and Kurz prove the existence of this refined equilibrium concept which is also known as unemployment equilibrium.

Dehez and Drèze (1984) extends the existence results for unemployment equilibrium by introducing price indexation instead of price boundaries as the cause of price rigidity. The proof of unemployment equilibrium's existence of van der Laan (1984) manages to combine Kurz and Dehez and Drèze. Weddepohl (1987) shows existence of unemployment equilibrium with linked price indexes. Herings (1996) on the other hand uses a generalized definition of rationings, and demonstrate that all known existence theorems follow as corollaries of his general result.

An interesting contribution is made by Drèze (1997) by showing that quantity constraints support disequilibrium even if prices are at their Walrasian market clearing values. The intuition for that result is that pessimistic expectations can sustain disequilibrium even at market clearing prices – a result undermining the coordination role generally attributed to prices. Herings (1998) shows that disequilibrium allocations constitute a connected set – a result seemingly independent of Drèze (1997). Nevertheless, the main messages of these two papers are juxtaposed by Citanna et al (2001) who show multiplicity of disequilibrium allocations at given prices (including Walrasian prices). They interpret multiple disequilibrium even at a unique Walrasian equilibrium prices as a problem of coordination as Drèze (1997).

The generalized rationings approach of Herings (1996) is carried to a more abstract level by Talman et al (2001) invoking a fixed point theorem of Browder (1960). In this way, they can prove existence where the set of admissible prices is any convex compact set – a substantial generalization in the literature. That paper also includes an extensive list of references.

1.5.3. Labor in non-Walrasian literature

Hitherto discussed assumptions are not pertinent to the primitives of an economy. In this regard, the non-Walrasian literature exhibits a stringent convention. Positing convex and closed preferences is virtually universal in the literature. However, our main concern is a set of common assumptions in the literature that impede introducing labor, and production to the economy. Now we turn to these conditions that preclude the notion of labor.

First of all, an omnipresent proviso in the literature is unbounded consumption sets in all dimensions. Second, assuming that all individuals own strictly positive amount of all goods as initial endowments is a common practice. As we shall see now, these two conditions are inconsistent with the notion of labor despite the non-Walrasian theory's original motivation of elucidating unemployment.

In virtue of the first condition of unbounded consumption sets in all dimensions, a wealthy consumer can buy someone else's free time and enjoy it as her own leisure. Thus, any pauper can sell her free time to affluent consumers who would enjoy their purchase as leisure. This is evidently incongruent with the notion of labor. This problem could be avoided by postulating an individual cannot enjoy leisure more than her labor endowment. However, this requires bounding the consumption set in the labor dimension. For an example of a careful and proper approach to introducing labor in this way see Moore (2007, p.86).

According to the second condition, which is known as interior endowments condition, each individual possesses from all goods in strictly positive amounts. This implies that everyone can supply all types of goods at all points in time and space. These goods also include all possible different types of labor. Hence, for example, everyone should be a pilot and a doctor at the same time according to interior endowments condition. More generally, interior endowments condition implies that everyone is an omnipotent consumer who is able to perform all professions simultaneously. But this is astoundingly unrealistic, and again incongruent with the notion of labor.

On top of that, interior endowments condition also guarantees irreducibility. Thus, all known existence conditions of Drèze equilibrium are stronger than all known existence conditions of Walrasian equilibrium (e.g. compare to McKenzie (1959) and Arrow and Hahn (1971)). Consequently, the most significant obstacle to invoke Drèze equilibrium to analyze disequilibrium due to non-existence of competitive equilibrium is arguably the condition of interior endowments.

1.5.4. Non-Walrasian equilibrium and non-existence of Walrasian equilibrium

In light of the above, we reach to the verdict that non-Walrasian analysis is not competent, yet, to introduce labor into economies that are in disequilibrium. Nor could the non-Walrasian techniques – given the current state of the literature- be used to scrutinize and elucidate how markets operate when there is no competitive equilibrium due to violation of irreducibility.

In order to surmount these problems, it is necessary to extend Drèze equilibrium's existence to economies with bounded consumption sets and boundary distribution of initial resources. When this is the case, labor can be introduced into the economy and economies without any Walrasian equilibrium can be analyzed based on the non-Walrasian principles.

1.6. Disequilibrium welfare analysis

As we argued above, voluntariness is germane to how markets operate, and often deemed a defining tenet of market games in economic theory. But voluntariness also proved to be a crucial condition regarding the efficiency properties of allocation mechanisms. The relation between voluntariness and efficiency, as we shall see, provides an effective toolbox to conduct disequilibrium welfare analysis. This section presents a survey of this strand of literature, namely the non-Walrasian efficiency analysis.

Silvestre (1985) proves that under certain conditions an allocation is Walrasian if and only if it is voluntary and efficient. This equivalence ensures that voluntary trade in non-clearing markets is inefficient. Moreover, not only does this result imply that disequilibrium is inefficient but also it is the most general result in disequilibrium welfare analysis.

Examples of other papers that address the efficiency properties of disequilibrium allocations are Nayak (1980), Schmeidler (1982), Madden (1982), Maskin and Tirole (1984), and Herings and Konovalov (2009, Proposition 3.3, p47). Nonetheless, as we mentioned above, all results that demonstrate the inefficiency of disequilibrium are special cases of Silvestre (1985). Hence, we only reflect upon Silvestre's equivalence.

Despite the common view among economists the inefficiency of disequilibrium allocations is not unconditional. Indeed, Silvestre stipulates convexity, smoothness, zero profits, and interior consumption, that is, strictly positive consumption of all goods by all individuals. If profits are positive, then the result still holds if strong voluntariness is employed in lieu of voluntariness. A voluntary allocation is strongly voluntary if there is at least one good (e.g. money) with a clear market.

Based on these two results one can conclude that actual markets are perpetually inefficient for non-clearing market phenomena is the norm. We find this conclusion very important. However, interior consumption condition is a stringent assumption which vitiates deducing that real world disequilibrium is inefficient. Moreover, Silvestre contrives an example where his results fail when all his assumptions but interior consumption hold. Therefore, there is no general result to test the efficiency of disequilibrium when consumption is on the boundary.

However, strictly positive consumption of all goods (i.e. interior consumption) is simply impossible. To see this, define goods finely so that commodities are distinguished over time and space. In this case, interior consumption implies simultaneous consumption at two distinct places. Furthermore, as we discussed previously, consuming someone else's time as leisure is inconsistent with the notion of labor. This also induces pervasive boundary consumption – as opposed to interior consumption – in the space of all labor types.

The unrealistic nature of interior consumption is well understood by the careful practitioners of the field of general equilibrium theory. For example, Moore (2007) states that "if we define commodities finely ... all reasonable allocations would result in each consumer's commodity bundle being on the boundary of its consumption set" and "consequently, even though we tend to think of boundary values as being a very special case, they are the norm in reality. (p. 222)" A similar remark is made by Geanakoplos (2004): "the more finely the commodities are described, the less likely are the commodity markets to have many buyers and sellers (p.116)." Arrow (1952) also notes that:

"Indeed, for any one individual, it is quite likely that the number of commodities on the market of which he consumes nothing exceed the number which he uses in some degree (p. 509)."

Moreover, boundary consumption is becoming more relevant due to the constant expansion of product spectrum over time.

Nevertheless, welfare analysis of disequilibrium without interior endowments condition is non-existent despite the fact that non-clearing markets are the norm in all modern economies. Silvestre (p. 813) concedes that interior consumption is very strong, and conjectures that it may be relaxed to the following weaker version: "each individual consumes some amount of what she owns initially". For example, workers rest for some time. Obviously, this proviso is far more plausible and weaker than interior consumption condition.

However, it is easy to show that this condition could not be sufficient due to the example in Arrow (1952) - the first rigorous example of Walrasian equilibrium's non-existence. Indeed, not only does this example demonstrate a case where there is no competitive equilibrium but also a case where a Pareto-efficient initial endowment vector cannot be a competitive equilibrium. This is sufficient to conclude that in this example there is a voluntary and efficient allocation that is not Walrasian, to wit, the initial endowment vector itself.

In order to surmount the problem of non-existence in Arrow's example, as we have seen in the first section, the following irreducibility condition is employed in the Walrasian literature: "Each individual owns something that is also valuable to others." Since disequilibrium welfare analysis makes assumptions on consumption plans but not on initial endowments the counter part of this irreducibility in disequilibrium welfare analysis may be the following: "Each individual consumes something that is also valuable to others." If this assumption is satisfied Silvestre's conjecture may turn out to be true. However, Silvestre's conjecture, to the best of our knowledge, has not been analyzed yet, and hitherto it is neither proved nor disproved.

CHAPTER 2

Too Many Poor But Very Few Rich

2.1. Introduction

Persistent unemployment and underutilized capacities are clearly observed in all market economies (Arrow (2005, p.15)). Motivated by this palpable and ubiquitous disequilibrium phenomenon, we develop a model where markets do not clear at any price due to unequal distribution of resources.

In a nutshell, we show that if low real wages clear labor markets in economies with high technology then the gap between the production capacity and the consumption of the poor workers cannot be filled by the rich property owners. For the consumption capacity of the rich cannot not be unlimited for all commodities. On the other hand, higher real wages raise labor costs, and induce unemployment. Hence our primary result ensues: markets do not equilibrate at any price.

Of course, consumers and firms in real life trade in markets subject to some prices, be it market clearing or not. Therefore we then ask: how does the nonexistence of market clearing prices effect voluntary trade? The answer is unequivocal in terms of efficiency: if market clearing prices do not exist then all voluntary allocations at all prices are Pareto-inefficient.[1] Moreover, for any voluntary allocation there is a Pareto-superior allocation that can be supported as a competitive equilibrium with an appropriate redistribution of income. We also prove that these efficiency enhancing redistributions are necessarily income transfers from property owners to workers.

What is the relevance of these results to observed disequilibrium in real life? Our findings suggest that a commonly purported solution to disequilibrium, to wit, price flexibility may be unsuccessful as follows. Wages in flexible labor markets would be low inducing inadequate demand for certain goods. However higher wages would give rise to unemployment. But resources would be allocated wastefully in either case. In other words, economic activity in free markets - that is, voluntary

[1]An allocation is voluntary at given prices if no agent benefits from trading less and consumers satisfy their budget constraints. For example, workers selling their labor voluntarily cannot improve their well-beings by working less. We interpret voluntary trade as economic acitivity in markets without government intervention.

allocations - is inefficient since prices cannot maintain market clearing. Nonetheless, in principle, an elaborate redistribution policy (e.g. welfare state policies) can solve the problem, and promote efficiency by stimulating demand through taxing the rich property owners and transferring the tax revenues to the poor workers.

The message that we strive to convey is very similar to the one of Brown and Heal (1979, p.573): "It is necessary to consider both the equity and the efficiency dimensions simultaneously."[2] More generally, the efficiency of markets is related to the initial distribution of resources which creates room for Pareto-improving redistribution. This observation is repeatedly demonstrated invoking somewhat standard imperfections such as externalities, incomplete markets, information problems, and nonconvexities (e.g. Geanakoplos and Polemarchakis (2008, 1986), Greenwald and Stiglitz (1986), Brown and Heal (1979)).

This paper reveals the surprising fact that this profound insight may still work in convex economies with complete markets, perfect information, and absence of externalities. Our argument is predicated upon the distinction of property owners and workers. A worker is defined to be a consumer who has only labor to sell in the market whilst property owners possess also non-labor resources. As Florig (2001a, 2001b) points out "[m]ost of the consumers have only labor to sell." That is, most of the consumers in real life are workers in our parlance. Indeed, existence of many workers and few property owners is the most salient condition to prove that markets do not clear at any price. Technically, this is equivalent to the nonexistence of competitive equilibrium *à la* Arrow-Debreu.

Chichilnisky (1995, p.80) remarks that: "The problem of nonexistence of a competitive equilibrium is pervasive. Despite the fact that market allocations are regarded as a practical solution to the resource allocation problem, many standard economies do not have a competitive equilibrium." In a similar vein, according to Rizvi (1991), generality of competitive equilibrium's existence is much-vaunted (see also Bryant (2010, 1997)). Nevertheless, there is an astounding paucity of formal analysis in this direction notwithstanding the significance of the subject.

This essay is hitherto the first study which provides general sufficiency conditions for the nonexistence of market clearing prices for convex production economies. The generality of our result suggests that our findings can be used to explain why markets do not clear - the central subject of the Keynesian tradition. For exchange economies a general condition of nonexistence of equilibrium is given by Chichilnisky (1995). Yet some simple examples are more widely known (e.g. Negishi (1987)). These are basically variants of Arrow (1951).

[2]We thank Graciela Chichilnisky for bringing Brown and Heal (1979) to our attention.

Likewise, we present the first scrutiny of how resources would be allocated when competitive equilibrium does not exist. As we stated above, the implications of this analysis run deep in terms efficiency and the role of redistributive policies in market economies. Another important insight of this essay, which is also congruent with the Keynesian tradition, is that redistribution yields Pareto-improvement only when the property owners are taxed and the workers are transfer receivers.

Our model satisfies all standard assumptions of convexity, closedness, and local non-satiation. However, we violate the condition of irreducibility - a proviso virtually present in all studies proving existence of equilibrium. An economy is irreducible when each agent possesses some initial endowments that can be used to make any other agent better-off at any feasible allocation.[3]

Therefore, economic theory is silent on the existence of competitive equilibrium when some consumers have only labor to supply and there is ample labor to utilize non-labor inputs at the full capacity. To see that, assume that an allocation with full capacity production is feasible. But increasing employment does not increase output at this allocation. Thus, labor cannot be used to improve the well-being of others. Hence, the economy fails to be irreducible, and none of the known results proving the existence of equilibrium applies.

Now we proceed with presenting a simple economy that is concocted to explain our results clearly as much as possible. This will be followed by a substantial generalization to underscore the pervasiveness of the problem.

2.2. An Example

Envisage a town where there are two produced goods: gold and bread. The inputs of gold mining and bread baking are labor and sector specific capital (i.e. machinery). Everyone strictly prefers possessing more gold to less.[4] However, an individual can eat bread up to a certain level of satiation.

To make things concrete, let

$$u_i : X_i \to \mathbb{R},\ x_i \mapsto u_i(x_i) = \sum_{h=1,2,3} x_{ih}^{\alpha},\ \alpha \in (0,1)$$

represent the preferences of individual i such that $X_i \subseteq \mathbb{R}_+^5$. Therefore, there are five goods in the economy. These are gold, bread, labor, mining machinery, and

[3] Several versions of this basic idea with technically subtle differences can be found in Moore (2006), Florig (2001b), Arrow and Hahn (1971), Debreu (1962), and McKenzie (1959).

[4] That is to say, gold is a desired good in our example. See Arrow and Debreu (1954, p.280) for a technical definition of desired goods.

bakery machinery respectively. Note that leisure is strictly preferred over working. No one enjoys consuming machinery of any kind.

A consumer is either a capitalist or a worker. The initial endowment of each worker i is

$$e_i = (0, 0, 1, 0, 0)$$

So a worker is a consumer who has to supply labor to buy bread and gold. The initial endowment of each capitalist i is

$$e_i = (0, 0, 1, m_1, m_2)\,.$$

In contrast to workers, each capitalist owns (m_1, m_2) units of machinery that are specific to mining and bakery. The numbers of the capitalists and the workers are K and W respectively.

It is implausible that a consumer could improve her well-being indefinitely by eating arbitrarily large amounts of bread. Eventually the point of satiation for bread should be relevant. Nor could anyone enjoy leisure more than her own labor-time. Thus we posit, $x \in X_i$ if and only if

$$x \geq 0 \text{ and } x_2 \leq \beta \text{ and } x_3 \leq 1.$$

Here β represents the satiation parameter for bread.[5] In the next section we will refer to such goods as "standard goods" - commodities subject to satiation.

Regarding production, we assume that $Y \subseteq \mathbb{R}^5$is the following simple fixed coefficient technology: Producing 1 unit of gold and 1 unit of bread requires 2 units of labor, a units of mining machinery and b units of bakery machinery. Formally, $y \in Y$ if and only if

$$\begin{aligned} y_1 + y_2 + y_3 &\leq 0 \\ ay_1 + y_4 &\leq 0 \\ by_2 + y_5 &\leq 0 \end{aligned}$$

where $(a, b) \gg 0$ is a given technology vector. Finally assume that bread baked utilizing all the machinery is enough to satiate the capitalists: $m_2 > b\beta$.

Write θ_i for the profit shares held by individual i. Note that Y is a constant returns to scale technology which makes the distribution of profits irrelevant. By the same token, any amount production is feasible with sufficient labor and machinery. However, inspection reveals that, taking the machinery fixed, production is subject to a full capacity constraint.

[5]We opt to put the satiation parameter in the consumption set. Instead, the idea that beer is subject to satiation could be represented in the rule of the utility function as well. However, the distinction is purely stylistic, and has no material consequences regarding our purposes.

The environment is fully characterized by the vector of exogenous

$$\mathcal{E} := (\alpha, \beta, a, b, m_1, m_2, K, W)$$

which we call an economy. Any given economy $\mathcal{E}$ satisfies all the standard convexity and closedness conditions, and the preferences are obviously locally non-satiating due to insatiable taste for gold.[6]

Our first result states that markets do not clear at any price if the number of workers is high enough.

Proposition 1. *For any given $(\alpha, \beta, a, b, m_1, m_2, K)$ there is a number W^* such that $W > W^*$ implies that there is no competitive equilibrium for the economy $\mathcal{E}$.*

Proof. See Appendix A-1. □

What does this result say about the actual economic activity in this town? In order to answer this question, we assume that firms and consumers trade voluntarily subject to given prices. An allocation is voluntary if no agent benefits from trading less. For example, employing less labor does not increase profits or working less does not increase utility when trade takes place voluntarily. For a formal definition see Section 5.

Now we shall show that all voluntary allocations at all prices are Pareto-inefficient if there is no competitive equilibrium.[7]

Proposition 2. *Let ξ be a voluntary allocation for $\mathcal{E}$ at prices $p \gg 0$. If $W > W^*$ then ξ is Pareto-inefficient.*

Proof. See Appendix B-1. □

Until now, it is demonstrated that any voluntary form of trade is Pareto-inefficient given $W > W^*$. The natural question at this point whether there is room for some policies to improve over voluntary trade. The answer is affirmative.

Let $q \in \mathbb{R}^{W+K}$ be a redistribution policy such that $\sum q_i = 0$. The monetary income of each consumer is

$$pe_i + \theta_i py + q_i.$$

[6] Roughly speaking, a preference relation is locally non-satiating if and only if for any consumption bundle there is another strictly preferred bundle in the close neighborhood of the former bundle. See Moore (2007, p.70) for a formal definition.

[7] One may ask where the prices come from if they are not determined through market clearing. We do not specify how prices are determined to preserve the generality of our model and this is a safe practice since the result holds for all prices. That is, the following result is not sensitive to the choice of price determination mechanism. Further inquiry in this line seems to be fruitful for endogenizing price-making behavior.

Our next result states that for any Pareto-inefficient voluntary allocation there is a Pareto-superior allocation that can be supported as a competitive equilibrium with an appropriate redistribution of income. Furthermore, these redistribution policies should be transfers of income from property owners to workers. Formally,

Proposition 3. *Let $\mathcal{E}$ be given. Then any Pareto-efficient allocation ξ can be supported as a competitive equilibrium with some transfers q. If $W > W^*$ then*

$$\sum_{i \in W} q_i > 0 > \sum_{i \in K} q_i.$$

Proof. See Appendix C-1. □

These results hinge upon the following hypotheses: existence of (i) high number of workers, (ii) full capacity production, and (iii) a consumption good subject to satiation (i.e. bread). We believe these are economically substantive assumptions.

Now let us explain why these assumptions violate irreducibility which is necessary for the nonexistence of competitive equilibrium. High number of workers ensures that utilizing all existing machinery at the full capacity is feasible. But in this case labor cannot be used to improve the well-being of the capitalists since increasing output by further employment is infeasible. Thus the economy cannot be irreducible. On the other hand, that individuals satiate for bread is not pertinent to the violation of irreducibility.

The major defect of this example is that the number of dimensions of the ambient space (i.e. $\mathbb{R}^5$) is too small for introducing time by the method dating commodities. The example is therefore static. So it is not clear whether the result is robust in dynamic economies with accumulation.

The next section settles down this question by generalizing all our results to the n dimensional ambient space. We will not address dynamic economies specifically but $\mathbb{R}^n$ gives enough dimensions to date commodities if one attempts to introduce time, and therefore adjustment of capacity constraint through production of capital.[8] We also substantially generalize the space of preferences. Moreover, an abstract approach is pursued for defining the capacity constraint so that we can totally dispense with the fixed coefficient production technology. It turns out that the nonexistence of competitive equilibrium holds under quite general conditions. Now we turn to this generalization.

[8] Of course, since n is finite here, this approach is restricted to finite though arbitrarily large horizon.

2.3. The General Model

Let us consider a private ownership economy $\mathcal{E}$. Consumers, producers and goods are indexed by $i \in I$, $j \in F$ and $h \in G$ respectively. There are r consumers, m producers, and n goods.

Consumer i is characterized by a consumption set $X_i \subset \mathbb{R}^n_+$, a reflexive binary relation $\succsim_i$ on X_i, a vector of initial resources $e_i \in \mathbb{R}^n_+$ and a vector of shares $\theta_i \in \mathbb{R}^m_+$, with $\sum_i \theta_{ij} = 1$, for all j. For further reference, let $\succ_i$ designate the asymmetric part of $\succsim_i$for all i .[9] Let $\succsim_i$ be locally non-satiating for all i. Each firm j is characterized by a production set $Y_j \subset \mathbb{R}^n$.

The set of all possible prices is the n dimensional unit simplex, i.e. $\Delta := \left\{p \in \mathbb{R}^n_+ : \sum_h p_h = 1\right\}$. Given the prices $p \in \Delta$ and the production plan $y_j \in Y_j$, the profits of firm j is $\pi_j(p) = py_j$. Therefore the budget set of individual i is

$$\beta_i(p) := \{x_i \in X_i : px_i \leq m_i(p)\}$$

where

$$m_i(p) := pe_i + \sum_j \theta_{ij}\pi_j(p).$$

Now competitive equilibrium can be defined. Interpretations for the definition of competitive equilibrium and the above constituents of the economy $\mathcal{E}$ are skipped since they are standard.

Definition 1 (Competitive equilibrium)**.** *An allocation*

$$(x_1^*, ..., x_r^*, y_1^*, ..., y_m^*, p^*) \in \mathbb{R}^{n(m+r)} \times \Delta$$

is a competitive equilibrium for $\mathcal{E}$ iff
(i) $x_i \succsim_i x_i^*$ and $x_i \in \beta_i(p^*)$ implies $x_i^* \succsim_i x_i$ for $i \in I$.
(ii) $p^* y_j > p^* y_j^*$ implies $y_j \notin Y_j$ for $j \in F$.
(iii) $\sum_i x_i^* - \sum_i e_i - \sum y_j^* \leq 0$ and $p^*\left(\sum_i x_i^* - \sum_i e_i - \sum y_j^*\right) = 0$.

Individuals derive income by means of their initial endowments and profit shares. In actual economies, palpable differences between individuals in terms of these resources' distribution motivated the classical social taxonomy according to classes, i.e. capitalists, rentiers, and workers. A similar classification is provided in the sequel. To this end, the definition of labor - one of the central notions in this study - should be introduced. A good l is a type of labor-time if and only if it is a non-produced good, which can be hired by firms but cannot be traded between individuals. Formally:

[9]That is, for all $i \in I$ and $\forall (x_i, x_i') \in X_i \times X_i$, $x_i \succ_i x_i' \Leftrightarrow [x_i \succsim_i x_i'$ and not $x_i' \succsim_i x_i]$.

Definition 2 (Labor)**.** *Let L be the set of commodities such that if $l \in L$ then (i) $y_j \in Y_j$ implies $y_{jl} \leq 0$ for all j, and (ii) $x_i \in X_i$ implies $x_{il} \leq e_{il}$ for all i. If $l \in L$ then good l is said to be a type of labor-time (or just "labor").*

L is the set of all labor types. Let the cardinality of L be ℓ. The first condition prescribes the types of labor to be only an input but not an output whereas, according to the second condition, an individual can enjoy only her own labor as leisure. The justification of the former condition is obvious, and needs no further explanation. The latter condition avoids pathological cases in which some individuals buy someone else's labor-time in order to enjoy it as leisure (see Moore (2007, p.86)).

Given the definition of labor, the definition of a worker is as follows. An individual is said to be a worker if and only if her monetary income is equal to the monetary value of her labor endowments.[10]

Definition 3 (Worker)**.** *Let W be the set of individuals such that if $i \in W$ then $m_i(p) = \sum_{l \in L} p_l e_{il}$ for all $p \in \Delta$. If $i \in W$ then individual i is said to be a worker. If $i \notin W$ then individual i is said to be a property owner and $i \in K$.*

W is the set of all workers while K is the set of all property owners. Workers, who possess labor type l, are designated by $W^l := \{i \in W : e_{il} > 0\}$ with the cardinality w_l. We also write $w := (w_1, ..., w_\ell) \in \mathbb{N}^\ell_{++}$.

The following notation will prove to be convenient for the rest of the paper:

$$x_i[\lambda, h] := x_i + \lambda v_h$$

where λ is a scalar, and v_h is the h^{th} unit coordinate vector.[11] Hence $x_i[\lambda, h]$ designates x_i with λ amount of additional consumption of good h for individual i.

Next, the notion of standard good is introduced. Good h is a standard good for a property owner $i \in K$ if and only if there is a treshold level such that consuming more of good h than this treshold does not contribute to the welfare of the property

[10] Note that being a worker does not inhibit the individual from enjoying the returns of her savings. That is because, savings do not qualify as capital endowment or profit shares, but as an integral of a consumption stream, which is financed by foregoing other consumption activities, in the parlance of general equilibrium theory.
However, it is implausible to think that a worker can earn large amounts of capital income trough the conduit of savings. Hence, in real life, a typical worker, who fits into our definition accuretely, would have no or little savings, and majorly live on wage income. As of 2006, the sum of capital and business income comprises less than %8 of income of %80 of the households in the US (see Mishel et al. (2009, p.82)).

[11] The vector of zeros except the h^{th} coordinate, which is equal to 1.

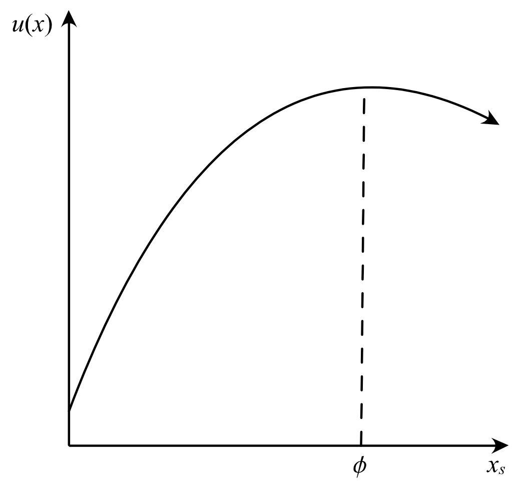

Figure 2.1. A typical graph of a utility function $u(\cdot)$ as a function of a standard good s while the consumption of other commodities is fixed.

owner $i \in K$. In other words, standard goods are commodities that are subject to satiation. Formally,

Definition 4 (Standard good)**.** *For each property owner $i \in K$ let $S_i \subset G$ be the set of goods such that if $s \in S_i$ then*

$$(\exists \phi\ > 0)\,(\forall x_i \in X_i) : \ x_{is} \geq \phi \text{ implies } x_i\,[\phi - x_{is}, s] \succsim_i x_i.$$

If good $s \in S_i$ then s is said to be a standard good for individual i.

If a produced good's consumption activity requires devotion of time then its serviceable consumption is limited (i.e. these goods are standard) since time is a limited resource. Obviously, many produced goods - arguably, most of the final consumer commodities - fall into this category. Another important reason for commodities to be subject to satiation - that is, being a standard good - is biological consumption capacity.

Note that if $\succsim_i$ is locally non-satiating then $s \in S_i$ implies that the optimal consumption of good s by property owner i is bounded above in $p \in \Delta$ if $p_s > 0$. That is to say, property owners' consumption demand for standard goods is limited. Hereafter, $\phi\,(i, s)$ stands for the least upper bound - which exists for real numbers are complete - of consumption of good $s \in S_i$ by individual $i \in I$. The set of non-labor standard goods common to all property owners is $S := \cap_{i \in K} S_i \backslash L$ which is assumed to be non-empty.

Now we will focus on demarcating the set of preferences of workers in our economy. Let $\succsim \subseteq X \times X$ where X is an arbitrary set. Then 'characteristics' is a triple $c := (X, \succsim, e)$ where e is a vector that will represent initial endowments. We will also use

$$c_i := (X_i, \succsim_i, e_i)$$

to designate the characteristics of individual i. The set of all characteristics is

$$\mathcal{C} := \left\{c : X \subseteq \mathbb{R}^n_+,\ \succsim \subseteq X \times X,\, e \in \mathbb{R}^n_+\right\}.$$

Let $C \subset \mathcal{C}$ be a given subset. We assume that C is the space of all possible worker characteristics. Formally, $i \in W$ implies $c_i \in C$. By imposing conditions on C, we will ensure that the preferences of workers satisfy certain assumptions. The first assumption is quite standard: Convexity and closedness (i.e. continuity). However, introducing convex and closed preferences rigorously requires defining the upper contour set

$$P^c(x) := \{q \in X : q \succsim x\} \text{ where } c \in \mathcal{C}.$$

The assumption is as follows:

Assumption 1. *Let $c \in C$. Then $P^c(x)$ is convex and closed for all $x \in X$.*

In the sequel we introduce a condition related to labor supply of workers. However, formulating this condition entails defining the open ball in the n-dimensional Euclidean space and the normal cone of a closed convex set.

Definition 5 (Open ball)**.** *Let $\delta > 0$ and $a \in \mathbb{R}^n$. Then the open ball with radius δ is*

$$B_\delta(a) := \{x \in \mathbb{R}^n : \|x - a\| < \delta\}$$

where $\|\cdot\|$ is the Euclidean norm.

Definition 6 (Normal cone)**.** *Let $X \subset \mathbb{R}^n$ be a closed and convex set with $x \in X$. Then the normal cone of X at x is*

$$N_x(X) := \{p \in \mathbb{R}^n : p(x - x') \leq 0 \textit{ for all } x' \in X\}.$$

Notice that if Assumption 1 holds then that $x \in X_i$ is an optimal consumption bundle for individual $i \in W$ at prices $p \in \Delta$ implies $p \in N_x(P^{c_i}(x))$. Now all the necessary tools to introduce the condition mentioned above are acquired.

Envisage an economy and an allocation such that workers - individuals who live on wage income - consume most of their labor endowments as leisure, and buy very little amount of all other goods. That is to say, workers in this economy voluntarily exchange their labor with produced goods in very little amounts. We will assume that this is only possible when the real wage rates are actually low, at least in terms of some standard goods. The assumption is expressed formally as follows.

Assumption 2. $(\forall \epsilon > 0)(\exists \delta > 0)(\forall c \in C)(\forall l \in L)(\forall x \in B_\delta(e))$ *:*

$$\rho \in N_x(P^{c_i}(x)) \text{ implies } \epsilon \geq \frac{\rho_l}{\rho_h}$$

for some $h \in S$*.*

As stated above, Assumption 2 says that if workers are willing to exchange their labor with small amounts of produced goods then real wages are low in terms of some standard goods. For smooth preferences, the Inada condition is typically sufficient to satisfy this assumption. The following remark attempts to clarify this aspect.

Remark 1. *Let* C *be a singleton. That is to say, all workers have identical characteristics. Assume that the unique element* $c \in C$ *can be represented by* $u : X \to \mathbb{R}$*. Let good* 1 *be labor and good* 2 *be a standard good. One can show that Assumption 2 is satisfied if*

$$x \to e \text{ implies } \frac{\partial u/\partial x_1}{\partial u/\partial x_2}(x) \to 0.$$

For example, $u(x) = \sum x_h^{1/2}$ *and* $X = [0,1] \times [0,\phi] \times \mathbb{R}_+^{n-2}$ *such that* $\phi > 0$*.*

The following assumption is a mild desirability condition. It says that individuals prefer leisure over working, which avoids strictly positive labor supply when wages are zero.

Assumption 3. $(\forall c \in C)(\forall l \in L)(\forall x \in X)(\forall \lambda < 0)$:

$$x[\lambda, l] \geq 0 \text{ implies } x \succ x[\lambda, l].$$

The production technology is assumed to involve capacity constraints. To formalize this property, we need to define the set of efficient production vectors given by,

$$\overline{Y}_j := \{y \in Y_j : \text{if } y' \geq y \text{ then } y' \in Y_j \text{ implies } y' = y\}.$$

Also define

$$\mathring{Y} := \left\{ y \in \sum_{j \in F} \overline{Y}_j : h \notin L \text{ implies } y_h + \sum_{i \in K} e_{ih} \geq 0 \right\}$$

which is the set of efficient production vectors that are feasible in terms of non-labor resources.

Assumption 4. *There is* $\overline{y} \in \mathring{Y}$ *such that* $y \in \mathring{Y}$ *implies* $|\overline{y}| \geq |y|$*.*

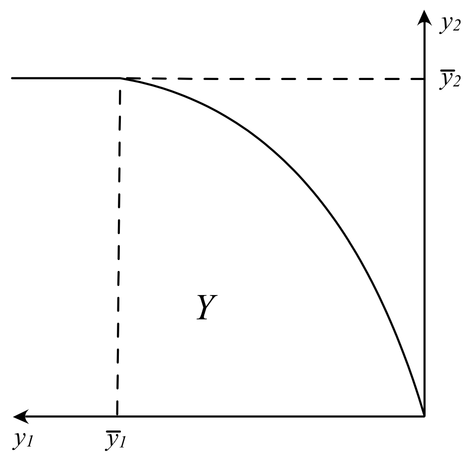

Figure 2.2. A production set with a capacity constraint.

Assumption 4 says that feasible output has a maximum with respect to labor given non-labor input. The market value of this maximum level of production is simply the definition of U.S. Census Bureau for full production capacity which is estimated quarterly.[12] Thus we maintain that Assumption 4 is realistic.

Note that this assumption does not impose a fixed capacity constraint. In particular, production of capital and adjusting capacity over time is not ruled out. For a comprehensive discussion of how general equilibrium models capture the notion of time see Moore (2007, p.359).

The next assumption requires that production technology is sufficiently high so that the upper bound of property owners' total demand for all $s \in S$ is less than the full capacity production.

Assumption 5. $(\forall s \in S) : \overline{y}_s > \sum_{i \in K} \phi(i, s) - e_{is}$.

The next item is our final assumption, which says that consumers do not enjoy consuming non-labor input (e.g. machinery, unprocessed ore, raw material, intermediate goods, etc.). But first we need to define the set of all characteristics of all individuals:

$$\overline{C} := \{c_i \in \mathcal{C} : c_i \notin C \text{ implies } i \in K\}.$$

The assumption is as follows.

[12]The U.S. Census Bureasu defines full production capacity as "[t]he market value of the maximum level of production ... fully utilizing the machinery and equipment in place."

Assumption 6. $\left(\forall c \in \overline{C}\right)(\forall x \in X)(\forall h \in G \backslash L)(\forall \lambda > 0)$:

$$\overline{y}_h < 0 \textit{ implies } x \succsim x[\lambda, h].$$

This condition ensures non-labor input endowments are supplied inelastically by households. The next section presents the result of nonexistence of competitive equilibrium.

2.4. Nonexistence of Equilibrium

Now we will show that, under the assumptions we discussed, economic activity can be coordinated through the principles of perfect competition only if the number of workers is low enough. In particular, for any given set of property owners, production technology, and set of worker characteristics, which satisfy Assumption 1-6, high number of workers implies that there is no competitive equilibrium.

So let us review these six assumptions very briefly. We assume that (i) preferences are convex, (ii) the Inada condition is satisfied by the preferences, (iii) there is disutility of working, (iv) feasible output has a maximum with respect to labor given non-labor input resources, (v) productivity for standard goods is high, and (vi) non-labor inputs are not consumption goods.

Theorem 1. *Let the set of property owners $(X_i, \succsim_i, e_i, \theta_i)_{i \in K}$, the production technology $(Y_j)_{j \in F}$, and the space of characteristics of workers C, which satisfy A1-A6, be given. Then there is a vector $\omega \in \mathbb{N}^{\ell}_{++}$ such that if the set of workers $(X_i, \succsim_i, e_i, \theta_i)_{i \in W}$ satisfies $w > \omega$ then there is no competitive equilibrium for the economy*

$$\mathcal{E} = \left((X_i, \succsim_i, e_i, \theta_i)_{i \in K \cup W}, (Y_j)_{j \in F} \right).$$

Before presenting the proof, let us expound it briefly. First, we will demonstrate that, assuming the claim is not true, high number of workers implies real wages are low (i.e. "cheap labor" or "poor workers") at least in terms of some standard goods in competitive equilibrium. This induces two irreconcilable results. According to the first one, cheap labor keeps the supply of standard goods very close to full capacity. According to the second one, poverty of workers keeps the demand of standard goods very low. Hence labor market clearing induces a glut in markets of standard goods, which means that market clearing prices do not exist.

Now we can present the proof formally:

Proof. Suppose the claim is not true. Then there is a production technology $(Y_j)_{j \in F}$, and a set of property owners $(X_i, \succsim_i, e_i, \theta_i)_{i \in K}$, and a set of characteristics of workers C, which satisfy A1-A6, such that for any $(\omega_1, ..., \omega_\ell)$ there is a set

workers $(X_i, \succsim_i, e_i, \theta_i)_{i \in W}$ subject to the following properties
(i) $(w_1, ..., w_\ell) > (\omega_1, ..., \omega_\ell)$
(ii) $c_i \in C$, $\forall i \in W$
(iii) competitive equilibrium for the economy

$$\mathcal{E} = \left((X_i, \succsim_i, e_i, \theta_i)_{i \in K \cup W}, (Y_j)_{j \in F} \right)$$

exists.

Now pick such $(Y_j)_{j \in F}$ and $(X_i, \succsim_i, e_i, \theta_i)_{i \in K}$ and C. Consider a sequence $\omega^t = (\omega_1^t, ..., \omega_\ell^t) \in \mathbb{N}_{++}^\ell$, $t = 1, 2, ...$, such that $\omega_l^t \to \infty$ as $t \to \infty$ for all $l \in L$. By assumption, this sequence gives rise to a sequence of economies $\mathcal{E}^t$ with a corresponding well-defined competitive equilibrium at each $t \in \mathbb{N}_+$. This sequence of competitive equilibria is given by $(x_1^t, ..., x_r^t, y_1^t, ..., y_m^t, p^t)$, $t \in \mathbb{N}_+$.

$\mathcal{E}^t$ contains a set of workers $\left(X_i^t, \succsim_i, e_i^t, \theta_i^t\right)_{i \in W^t}$ at each $t \in \mathbb{N}_+$. Let W^t designate the index set of all workers at each $t \in \mathbb{N}_+$. Also let $W^{l,t}$ denote the index set of workers who possess labor good $l \in L$ whereas w_l^t is the cardinality of $W^{l,t}$. Moreover $(w_1^t, ..., w_\ell^t) > \omega^t$ must hold for all $t \in \mathbb{N}_+$.

Since we will frequently use the market clearing condition, let us state it separately:

$$\sum_{j \in F} y_{jh}^t - \sum_{i \in W^t \cap K} x_{ih}^t + \sum_{i \in W^t \cap K} e_{ih}^t \leq 0 \text{ for all } h \in G \tag{2.1}$$

which holds with strict inequality only if $p_h = 0$.

Define

$$\mu_{hl}(p) := \sup \left\{ x \in \mathbb{R}_+ : p_l \geq p_h x \right\}.$$

One can interpret $\mu_{hl}(p)$ as the real wage of labor l with respect to good h. Now it will be shown that there is $s \in S$ such that $\mu_{sl}(p^t) \to 0$ for all $l \in L$. In words, all workers' real wages in terms of some standard goods shrink to zero as the number of workers increases. To this end, we shall prove that for each type of labor $l \in L$ one can pick $\imath^t \in W^{l,t}$ at each t to construct a sequence of workers $\imath^t$, $t \in \mathbb{N}_+$, such that $\|x_{\imath^t}^t - e_{\imath^t}^t\| \to 0$. Otherwise there would be a good h such that

$$\sum_{i \in W^{l,t}} \left| x_{ih}^t - e_{ih}^t \right| \to \infty.$$

If $h \in L$ then

$$\sum_{i \in W^{l,t}} x_{ih}^t - e_{ih}^t \to -\infty$$

since

$$x_{ih}^t - e_{ih}^t \leq 0 \text{ for all } i \text{ and } h \in L$$

which follows from the Condition (ii) of the definition of labor. As a consequence, there is τ_1 such that $t > \tau_1$ implies $p_h^t = 0$ since

$$p_h^t > 0 \text{ implies } 0 \geq \sum_{j\in F} y_{jh}^t \geq \overline{y}_j.$$

But if $p_h^t = 0$ then

$$\sum_{i\in W^{l,t}} x_{ih}^t - e_{ih}^t = 0$$

due to Assumption (3). On the other hand, if $h \notin L$ then

$$\sum_{i\in W^{l,t}} \left|x_{ih}^t - e_{ih}^t\right| \to \infty \text{ implies } \sum_{i\in W^{l,t}} x_{ih}^t \to \infty$$

which contradicts feasibility due to Assumption (4). These show that for each type of labor $l \in L$ there is a sequence of workers $\imath^t$, $t \in \mathbb{N}_{++}$, such that$\|x_{\imath^t}^t - e_{\imath^t}^t\| \to 0$, which is equivalent to

$$(\forall\delta > 0)\,(\exists t_\delta) : t > t_\delta \Rightarrow \left\|x_{\imath^t}^t - e_{\imath^t}^t\right\| < \delta.$$

In this case, due to Assumption (2), for any $\epsilon > 0$, $\exists\ t_\delta$, such that for every $l \in L$, $t > t_\delta$ implies $\frac{p_l^t}{p_s^t} \leq \epsilon$ for some $s \in S$. Therefore, there is $s \in S$ such that $\mu_{sl}\,(p^t) \to 0$ for all $l \in L$. Let $\widehat{s} \in S$ be one of such standard goods. Note that one can take $p_{\widehat{s}}^t > 0$ for all t without loss of generality since $\mu_{\widehat{s}l}\,(p^t) \to 0$ for all $l \in L$.

Write $\widetilde{y}_h^t = \sum_{j\in F} y_{jh}^t$ where $h \in G$. Now we will prove that $\widetilde{y}_{\widehat{s}}^t \to \overline{y}_{\widehat{s}}$. Due to profit maximization

$$p_{\widehat{s}}^t\left(\widetilde{y}_{\widehat{s}}^t - \overline{y}_{\widehat{s}}\right) + \sum_{h\neq\widehat{s}} p_h^t\left(\widetilde{y}_h^t - \overline{y}_h\right) \geq 0.$$

Furthermore, if $p_h^t > 0$ then $\overline{y}_h \geq \widetilde{y}_h^t \geq 0$ for all h such that $\widetilde{y}_h^t \geq 0$ (Assumption (4)). This yields

$$p_{\widehat{s}}^t\left(\widetilde{y}_{\widehat{s}}^t - \overline{y}_{\widehat{s}}\right) + \sum_{h\in N^t} p_h^t\left(\widetilde{y}_h^t - \overline{y}_h\right) \geq 0.$$

where

$$N^t := \left\{h \in G : \widetilde{y}_h^t < 0\right\}.$$

Thus,

$$p_{\widehat{s}}^t\left(\widetilde{y}_{\widehat{s}}^t - \overline{y}_{\widehat{s}}\right) + \sum_{h\in N^t\backslash L} p_h^t\left(\widetilde{y}_h^t - \overline{y}_h\right) + \sum_{h\in L} p_h^t\left(\widetilde{y}_h^t - \overline{y}_h\right) \geq 0.$$

However,

$$\sum_{h\in N^t\backslash L} p_h^t\left(\widetilde{y}_h^t - \overline{y}_h\right) = 0.$$

To see this, observe that $p_h^t > 0$ implies $0 > y_h^t \geq \overline{y}_h\ \forall h \in N^t$ (Assumption 4). But $\widetilde{y}_h^t > \overline{y}_h$ yields

$$\widetilde{y}_h^t + \sum_{i\in K} e_{ih} > 0,\ \forall h \in N^t\backslash L$$

since

$$\overline{y}_h + \sum_{i \in K} e_{ih} \geq 0, \; \forall h \in N^t \backslash L$$

due to the definition of $\overset{\circ}{Y}$. Since non-labor input is not enjoyed by consumers (Assumption 6) that would give $p_h^t = 0$.

Hence,

$$p_{\widehat{s}}^t \left(\widetilde{y}_{\widehat{s}}^t - \overline{y}_{\widehat{s}} \right) + \sum_{h \in L} p_h^t \left(\widetilde{y}_h^t - \overline{y}_h \right) \geq 0.$$

Dividing both sides by $-p_{\widehat{s}}^t$ gives

$$\begin{aligned} 0 &\geq \overline{y}_{\widehat{s}} - \widetilde{y}_{\widehat{s}}^t + \frac{\sum_{l \in L} p_h^t \left(\overline{y}_l - \widetilde{y}_l^t \right)}{p_{\widehat{s}}^t} \\ &= \overline{y}_{\widehat{s}} - \widetilde{y}_{\widehat{s}}^t + \sum_{l \in L} \mu_{\widehat{s}l} \left(p^t \right) \left(\overline{y}_l - \widetilde{y}_l^t \right) \end{aligned}$$

Above inequality implies $\widetilde{y}_{\widehat{s}}^t \to \overline{y}_{\widehat{s}}$ since $\mu_{\widehat{s}l} \left(p^t \right) \to 0$ for all $l \in L$.

On the other hand, $\sum_{l \in L} \left| \overline{y}_l \right| \mu_{hl} \left(p^t \right)$ is the maximum amount of good h that all workers can afford at each t. Hence

$$\sum_{l \in L} \left| \overline{y}_l \right| \mu_{hl} \left(p^t \right) \geq \sum_{i \in W^t} x_{ih}^t \tag{2.2}$$

However, this ensures $\sum_{i \in W^t} x_{i\widehat{s}}^t \to 0$ since $\mu_{\widehat{s}l} \left(p^t \right) \to 0$. Consequently,

$$\overline{y}_{\widehat{s}} > \sum_{i \in K} \left(\phi \left(i, \widehat{s} \right) - e_{i\widehat{s}} \right) + \sum_{i \in W^t} x_{i\widehat{s}}^t$$

holds for all t with only finite exceptions since

$$\overline{y}_h > \sum_{i \in K} \left(\phi \left(i, h \right) - e_{ih} \right) \text{ for all } h \in S \text{ (by Assumption 5)}$$

and $\sum_{i \in W^t} x_{i\widehat{s}}^t \to 0$ holds. Nevertheless, it is already shown that $y_{j\widehat{s}}^t \to \overline{y}_{j\widehat{s}}$. As a result, the inequality

$$\sum_{j \in F} y_{j\widehat{s}}^t > \sum_{i \in K} \left(\phi \left(i, \widehat{s} \right) - e_{i\widehat{s}} \right) + \sum_{i \in W^t} x_{i\widehat{s}}^t \geq \sum_{i \in K \cup W^t} \left(x_{i\widehat{s}}^t - e_{i\widehat{s}}^t \right)$$

ensues for all t with only finite exceptions. Since p^t is a competitive equilibrium price vector, it must be the case that $p_{\widehat{s}}^t = 0$ for all t with finite exceptions. This contradicts $\mu_{\widehat{s}l} \left(p^t \right) \to 0$. □

2.5. Voluntary Allocations

In this section we ask the following question: if there is no competitive equilibrium how does the economy work? To answer this question we posit that firms and consumers trade voluntarily at given prices.

A feasible allocation is voluntary at given prices if no agent benefits from trading less and all consumers satisfy their budget sets. For instance, if firms and workers exchange labor and produced commodities voluntarily then workers cannot increase their utility by selling less labor. In a similar fashion, firms cannot increase profits by producing less. The standard interpretation of this condition is that no agent is forced to trade.

We maintain that unfettered markets without government intervention would satisfy voluntariness. Indeed, in economic theory, voluntariness is widely considered as a defining tenet of market mechanisms of any type (e.g. Beviá et al (2003), Svensson (1991), Benassy (1986)). Let us give the precise definition. But first the concept of feasibility:

Definition 7 (Feasibility). *An allocation* $\xi^* = (x_1^*, ..., x_r^*, y_1^*, ..., y_m^*) \in \mathbb{R}^{n(r+m)}$ *is an* $n-$*tuple of* r *consumption vectors and* m *production vectors;* ξ^* *is feasible iff* $x_i^* \in X_i$, $\forall i \in I$, $y_j^* \in Y_j$,$\forall j \in F$, *and:* $\sum_{i \in I} x_i^* - \omega_i \leq \sum_{j \in F} y_j^*$.

Voluntariness is defined as follows:[13]

Definition 8 (Voluntariness). *A feasible allocation* ξ^* *is voluntary for* $(\mathcal{E}, p)$ *iff: (i)* $\forall i \in C$, x_i^* *maximizes* $\succsim_i$ *subject to:* $x_i \in \beta_i(p)$, *and to:*

$$\min\{0, x_{ih}^* - \omega_{ih}\} \leq x_{ih} - \omega_{ih} \leq \max\{0, x_{ih}^* - \omega_{ih}\} \quad (j = 2, ..., n)$$

(ii) $\forall j \in F$, y_j^* *maximizes* py_j *subject to:* $y_j \in Y_j$, *and to:*

$$\min\{0, y_{ih}^*\} \leq y_{ih} \leq \max\{0, y_{ih}^*\} \quad (j = 1, ..., n)$$

The following assumption says that the economy $\mathcal{E}$ is smooth and convex.

Assumption 7. *(i) The boundary of* Y *is a smooth manifold and* Y *is a convex set. (ii) There is a smooth concave function* $u_i : X_i \to \mathbb{R}$ *representing* $(\succsim_i, X_i)$ *for all* i.

The next result states that any voluntary allocation is Pareto inefficient if market clearing prices do not exist in convex and smooth economies. Writing $\text{int}(Z)$ for the interior any set Z:

Theorem 2. *Let* $\mathcal{E}$ *and* C *and* $p \gg 0$ *be given. Assume A1-A7 and* ξ^* *is a voluntary allocation for* $(\mathcal{E}, p)$. *If* $w > \omega$ *and* $x_i^* \in \text{int}(X_i)$ *for all* i *then* ξ^* *is Pareto-inefficient.*

[13] Note that there is no quantitiy constraint for the consumption of good 1. This is justified by interepreting good 1 as a medium of exchange (e.g. money) in the literature.

Remark 2. *Recall that $w > \omega$ says the number of workers is sufficiently high so that there is no competitive equilibrium (see Theorem 1).*

Proof. Let $\mathcal{E}$ and C and $p \gg 0$ be given. Assume A1-A7 and ξ^* is a voluntary and Pareto-efficient allocation for $(\mathcal{E}, p)$. If $x_i^* \in \operatorname{int}(X_i)$ for all i then ξ^* is a competitive equilibrium. See Theorem 2 of Silvestre (1985)) for the proof. It is noteworthy that A1-A6 are not necessary up this point. However, A1-A6 implies that ω is well-defined. Thus, $w > \omega$ yields that there is no competitive equilibrium. Contradiction. □

2.6. Redistribution of Income

This section addresses the policy implications of the model. We show that if there is no competitive equilibrium then for any voluntary allocation there is a Pareto-superior allocation that can be supported as a competitive equilibrium with some lump-sum transfers. Moreover, these redistribution policies necessarily transfer income from property owners to workers.

Let $q := (q_1, ..., q_r) \in \mathbb{R}^r$ be a lump-sum transfer such that

$$\sum_{i=1}^{r} q_i = 0.$$

Now replace the monetary income of consumers with

$$m_i(p) := pe_i + \textstyle\sum_j \theta_{ij}\pi_j(p) + q_i \text{ for all } i$$

and keep all other components of the model unaltered.

Now we can show that the inconsistency of competitive equilibrium based on economic classes scrutinized heretofore can be remedied by an appropriate redistribution of income.

Theorem 3. *Let $\mathcal{E}$ and C which satisfy A1-A7 be given. Assume* $\operatorname{int}(\sum_i X_i) \cap \sum_i e_i + \sum_j Y_j \neq \varnothing$*. Then any Pareto-efficient allocation ξ^* such that* $x_i^* \in \operatorname{int}(X_i)$ *for all i can be supported as a competitive equilibrium with some lump-sum transfers q. If $w > \omega$ then*

$$\sum_{i \in W} q_i > 0 > \sum_{i \in K} q_i.$$

Proof. For the first part, the Second Fundamental Theorem of Welfare applies. See Moore (2007, p. 222) Theorem 7.38.

The proof of that any q which supports a competitive equilibrium is a transfer of income from property owners to workers is as follows. Suppose there is a production technology $(Y_j)_{j \in F}$, and a set of property owners $(X_i, \succsim_i, e_i, \theta_i)_{i \in K}$, and a set of

characteristics of workers C, which satisfy A1-A7, such that for any $(\omega_1, ..., \omega_\ell)$ there is a set workers $(X_i, \succsim_i, e_i, \theta_i)_{i \in W}$ subject to the following properties
(i) $(w_1, ..., w_\ell) > (\omega_1, ..., \omega_\ell)$
(ii) $c_i \in C$, $\forall i \in W$
(iii) competitive equilibrium for the economy

$$\mathcal{E} = \left((X_i, \succsim_i, e_i, \theta_i)_{i \in K \cup W}, (Y_j)_{j \in F} \right)$$

exists with a lump-sump transfer q.

(iv) $\sum_{i \in W} q_i \leq 0 \leq \sum_{i \in K} q_i$

Now pick such $(Y_j)_{j \in F}$ and $(X_i, \succsim_i, e_i, \theta_i)_{i \in K}$ and C. Consider a sequence $\omega^t = (\omega_1^t, ..., \omega_\ell^t) \in \mathbb{N}_{++}^\ell$, $t = 1, 2, ...$, such that $\omega_l^t \to \infty$ as $t \to \infty$ for all $l \in L$. By assumption, this sequence gives rise to a sequence of economies $\mathcal{E}^t$ with a corresponding well-defined competitive equilibrium at each $t \in \mathbb{N}_+$. This sequence of competitive equilibria is given by $(x_1^t, ..., x_r^t, y_1^t, ..., y_m^t, p^t)$, $t \in \mathbb{N}_+$.

$\mathcal{E}^t$ contains a set of workers $\left(X_i^t, \succsim_i, e_i^t, \theta_i^t\right)_{i \in W^t}$ and a lump-sum transfer q^t at each $t \in \mathbb{N}_+$. After replacing Eq (2.2) with

$$\sum_{l \in L} |\overline{y}_l| \, \mu_{hl} \left(p^t\right) + \frac{\sum_{i \in W} q_i^t}{p_h^t} \geq \sum_{i \in W^t} x_{ih}^t.$$

the proof of Theorem 10 still applies. However, this contradicts that q^t supports a competitive equilibrium at each t. □

2.7. Historical Remarks

Explaining why markets do not clear has ancient roots in economic thought. One early attempt in this direction is the theory of underconsumption which is our inspiration. Underconsumption theories can be traced back to Malthus and Sismondi, who are prominent proto-Keynesians objecting Say's law. Chipman (1965, p.707-714) illustrates their objection with an example of competitive equilibrium's nonexistence which is an exchange economy version of our example. This example is due to Arrow (1952, p.527).

As Arrow and Hahn (1971, p.347) and Bryant (2010, 1997) point out, the nonexistence of market clearing competitive prices is also germane to the original Keynesian view of markets' incompetency of self-regulation. In particular, that non-clearing market phenomena can be persistent even if there is no price rigidity is a famous theme in the Keynesian tradition. Our analysis can be used to reach this conclusion since even the most flexible price mechanism cannot equate demand and supply if there is no market clearing equilibrium.

This relation was noticed during the genesis of modern disequilibrium theory, and early attempts to formalize the Keynesian economics with flexible prices as the nonexistence of market clearing prices can be found in Klein (1947) and Patinkin (1948) even though they lack rigorous micro foundations. Arrow and Debreu (1954, p. 281) give a nonexistence example with proper micro foundations - alas in words - for a production economy. Although their example does not intend to theorize non-clearing market phenomenon, there is a clear reference to disequilibrium and structural unemployment. Indeed, our study can be deemed a generalization of this example.

Another historical example is Thornton, who contrives examples in which trade takes place at disequilibrium for there is no market clearing equilibrium (see Negishi (1986) for further details). Nevertheless, our result is closer to Malthus and Sismondi's underconsumptionist critique rather than Thornton's approach for Thornton invokes non-convexities which we do not.

CHAPTER 3

On the Existence of Drèze Equilibrium

3.1. Introduction

Non-clear market phenomenon is flagrant in market economies. A particular example is persistent unemployment which all market economies suffer. In a similar vein, palpable capacity underutilization, which is generally deemed to indicate lack of demand, is perpetual and widespread - so much so that it constitutes the norm. Thus elucidating the reasons and implications of disequilibrium is empirically relevant, and needless to say, normatively significant.

Non-Walrasian theory provides a general and technically elegant framework to analyze how markets operate when prices do not clear markets. This essay aims at generalizing existence results in this strand of general disequilibrium literature.

The primary impetus for this generalization is to surmount the following problem. Certain assumptions in the non-Walrasian literature exclude even the simplest and the most common type of worker-capitalist models. Construing these problematic assumptions properly unravels that the non-Walrasian tradition is actually inconsistent with the notion of labor. However, we show that these stringent assumptions, which induce exclusion of the notion of labor from the theory, can be dispensed with to a large extent, and the existence results for non-Walrasian equilibrium can be generalized appropriately.

Non-Walrasian equilibrium is defined as a voluntary allocation without any frictions. An allocation is voluntary if no agent can benefit from trading less subject to the budget constraint at the prevailing prices. On the other hand, absence of frictions prescribes that all mutually beneficial trade takes place in all markets.[1] It is easy to see that Walrasian equilibrium always satisfies these conditions but, of course, not the other way around.

Based on these defining tenets, there are several non-Walrasian equilibrium definitions, namely those of Drèze (1975), Benassy (1975,1976), and Malinvaud and Younès (1977). However Silvestre (1982, 1983) shows that existence of Drèze equilibrium, is sufficient for the existence of other types of non-Walrasian equilibria.

[1]Note that nonexistence of frictions is necessary for Pareto-efficiency, but not sufficient.

Therefore, we focus on Drèze equilibrium with production in this study. A precise definition of Drèze equilibrium is provided in the main text.

Talman et al (2001) and Herings (1996) provide extensive literature reviews for existence of Drèze equilibrium. First of all, stipulating convex and closed preferences are common in the literature. We also use these properties here. However, our main concern is a set of common assumptions in the literature that impede introducing labor, and production to the economy. Now we turn to these conditions that preclude the notion of labor.

First of all, it is remarkable that the literature of existence of Drèze equilibrium is confined to exchange economies. Indeed, the original definition of Drèze explicitly excludes production. Second, postulating unbounded consumption sets in all dimensions is omnipresent in the literature. Finally, positing that all individuals own strictly positive amount of all goods as initial endowments is very standard. We strongly maintain that these three conditions hinder introducing labor to the economy notwithstanding the non-Walrasian theory's original spur of understanding unemployment. Let us reflect upon why these three stipulations are inconsistent with labor.

That an exchange economy is inconsistent with the notion of labor due to absence of production is evident and needs no further comment. The second condition of unbounded consumption sets in all dimensions implies that there is no obstacle for a wealthy consumer to buy someone else's free time and enjoy it as her own leisure. Thus, an affluent individual can consume an underemployed individual's free time. This is also evidently incongruent with the notion of labor. In this regard, Moore (2007, p.86) gives an elaborate example for how to define labor properly.

The third condition, which is known as interior endowments, implies each individual can supply all types of goods (including different types of labor, if they exist) at all points in time and space. In other words, everyone owns labor endowments of all existing professions (e.g. pilot, coal miner, doctor, etc.). In fact these omnipotent individuals are able to perform all these professions simultaneously if interior endowments condition is satisfied. But this is astoundingly unrealistic, and again incongruent with the notion of labor.

It is noteworthy that interior endowments assumption is also germane to the existence of Walrasian equilibrium. For instance, the seminal paper of Arrow and Debreu (1954) postulates interior endowments condition whereby budget sets are ensured to be continuous. In fact, it is well known that budget sets exhibit a pervasive discontinuity problem without interior endowments (Chichilnisky (1995, p.80) and Arrow (1974, p. 267)). However, the Walrasian tradition, contrary to its non-Walrasian counterpart, acknowledges the fact that interior endowments condition is an unsatisfactory solution to the problem of budget set discontinuity. Consequently, there is a well-developed Walrasian equilibrium existence literature which majorly focuses on finding more realistic assumptions instead of interior endowments (e.g. Moore (2006), Florig (2001), Arrow and Hahn (1971), Debreu (1962), and McKenzie (1959)). Our study can be deemed the first step to address the same issue in the purview of non-clear markets.

In this regard, our existence covers production economies by invoking Silvestre's (1983) generalized definition of Drèze equilibrium. Moreover, we dispense with the unbounded consumption set condition. Hence, trade of labor-time among individuals as a consumption good can be avoided by stipulating that no individual can enjoy leisure more than the initial amount of the labor-time endowment that she initially has. As for the distribution of endowments, we posit a significantly weaker version of the interior endowments condition. The weaker version says that an individual can supply some strictly positive amount of all the goods that she possesses initially. For example, a worker, whose income is solely comprised of her wage, can supply some positive amount of labor, but not necessarily all existing goods as interior endowments assumption requires.

Under these conditions our main result is the lower hemi-continuity of budget correspondences. The assumptions that are discussed until now are only pertinent to this continuity result. Then we demonstrate how this result can be exploited to prove the existence of Drèze equilibrium with production.

The next section introduces the notation.

3.2. Notational framework

Let us consider a private ownership economy $\mathcal{E}$ with r consumers (1 to r), $m-r$ producers ($r+1$ to m) and n commodities, indexed respectively $i=1,...,r$, $j=r+1,...,m$ and $h=1,...,n$. Consumer i is characterized by a consumption set $X_i \subset \mathbb{R}^n_+$ on which her utility function $u_i : X_i \to \mathbb{R}$ is defined, a vector of initial resources $\omega_i \in \mathbb{R}^n_+$ and a vector of shares $\theta_i \in \mathbb{R}^{m-r}_+$, with $\sum_i \theta_{ij} = 1$, $j=r+1,...,m$. Each firm j is characterized by a production set Y_j.

A price system is a vector $p \in P$ where

$$P := \left\{p \in \mathbb{R}^n_+ : \overline{p} \geq p \geq \underline{p}\right\}$$

such that $\overline{p} \in \mathbb{R}^n_{++}$ and $\underline{p} \in \mathbb{R}^n_{++}$. Note that P is non-empty if and only if $\overline{p} \geq \underline{p}$.Given the prices $p \in P$ and the production plan $y_j \in Y_j$, the profits of firm j is $\pi_j(p) = py_j$. A rationing scheme is a set of vectors $(L,l) \in \mathbb{R}^n_+ \times \mathbb{R}^n_-$, which are absolute constraints on individual net trades. Therefore the budget set of individual i is

$$\beta_i(p, L, l) := \left\{ x_i \in X_i : px_i \leq p\omega_i + \sum_j \theta_{ij}\pi_j(p),\ L \geq x_i - \omega_i \geq l \right\}$$

so that individuals choose financially feasible consumption bundles, which respect quantity restrictions in the economy. Regarding the consumption sets, we posit the following assumptions for all consumers.

Assumption 8. *X_i is convex*

Assumption 9. *There exists $\underline{x}_i \in X_i$. such that (i) $\underline{x}_i \leq \omega_i$, (ii) $\underline{x}_i \leq x \leq \omega_i$ implies $x \in X_i$, and (iii) $\omega_{ih} > 0$ implies $\omega_{ih} > \underline{x}_{ih}$ for all h $(h = 1, ..., n)$.*

Note that Assumption 8 and 9 subsume $\omega_i \in \text{int}\, X_i$, and $X_i \subset X_i + \mathbb{R}^n_+$, which are standard in the non-Walrasian literature (e.g. Talman et al (2001)). As we discussed earlier, these stringent assumptions exclude economies with labor. As for the preferences, we assume:

Assumption 10. *u_i is continuous and quasi-concave.*

The following assumptions hold for all firms.

Assumption 11. *Y_j is closed and convex and $0 \in Y_j$.*

We also stipulate that the total production set $Y = \sum_j Y_j$ satifies:

Assumption 12. *Y is closed such that $Y \cap (-Y) \subset \{0\}$ and $\mathbb{R}^n_+ \subset -Y$*

According to this assumption the total production set is closed, irreversible, and satisfies free-disposability condition, whereby we compactify the economy.

Now the definition of Drèze equilibrium can be introduced:

Definition 9. *An allocation $z^* \in \mathbb{R}^{nm}$ is a Drèze equilibrium for $\mathcal{E}$ if there exist restrictions $(L, l) \in \mathbb{R}^n_+ \times \mathbb{R}^n_-$ and a price system $p \in P$ such that*
(i) $z^*_i + \omega_i$ maximizes u_i on the budget set $\beta_i(p, L, l)$ for $i = 1, ..., r$.
(ii) z^*_j maximizes $-pz_j$ subject to $-z_j \in Y_j$ and $z_j \in [l, L]$ for $j = r + 1, ..., m$.
(iii)$\sum_{a=1,...,m} z^*_a = 0$.
(iv) If there is an agent a, $L_h = z^*_{ah}$ then for all agents a', $z^*_{a'h} > l_{a'h}$ and; if there is an agent a, $l_{ah} = z^*_{ah}$ then for all agents a', $z^*_{a'h} < L_{a'h}$ for $h = 1, ..., n$.
(v) $\overline{p}_h > p_h$ implies $L_h > z^*_{ah}$, and $\underline{p}_h < p$ implies $l_h < z^*_{ah}$ for $a = 1, ..., m$.

The conditions (i) and (ii) are consumers' and firms' optimizing behavior, which are generalized with respect to Walrasian counterparts via introducing quantity constraints. The condition (iii) is equality of total demand and total supply. The condition (iv) is known as "frictionless markets condition", which guarantees that supply and demand of a particular good cannot be rationed simultaneously. The final condition ensures price flexibility in the interior of P.

In the sequel, two important continuity results, and the proof of Drèze equilibrium's existence are presented.

3.3. Existence of Drèze equilibrium

The following result shows that budget sets are continuous in $p \in P$, quantity restrictions $(L, l) \in \mathbb{R}^n_+ \times \mathbb{R}^n_-$, and profit income $\pi \in \mathbb{R}_+$. Given that the initial endowments are fixed, the result is a generalization of Theorem 2.2 of Herings (1996). We omit the subscript for individuals for notational clarity.

Theorem 4. *Let Assumption 1 and 2 hold.Then the correspondence* $\beta : P \times \mathbb{R}^n_+ \times \mathbb{R}^n_- \times \mathbb{R}_+ \to \mathbb{R}^n$ *given by*

$$\beta(p, L, l, \pi) = \{x \in X : p(x - \omega) \leq \pi,\ L \geq x - \omega \geq l\}$$

is continuous in (p, L, l, π).

Proof. First of all, $\omega \in \beta(p, L, l, \pi)$ for all (p, L, l, π) since $\omega \in X$, and $L \geq 0 \geq l$. Hence $\beta(p, L, l, \pi)$ is non-empty for all (p, L, l, π). Now we need to show that $\beta(p, L, l, \pi)$ is lower amd upper hemi-continuous.

Regarding lower hemi-continuity, consider $(p^t, L^t, l^t, \pi^t) \to (p, L, l, \pi) \in P \times \mathbb{R}^n_+ \times \mathbb{R}^n_- \times \mathbb{R}_+$, and let $x \in \beta(p, L, l, \pi)$. We need to show there is a sequence x^t in $\beta(L^t, l^t, \pi^t)$ such that $x^t \to x$.

To avoid cumbersome notation, let $\beta^t = \beta(L^t, l^t, \pi^t)$. Now define

$$\begin{aligned} J &:= \{j : x_{j-}\omega_j > 0\} \\ H &:= \{j : x_{j-}\omega_j < 0\} \\ K &:= \{j : x_{j-}\omega_j = 0\} \end{aligned}$$

Let $\mu^t_j := L^t_j / (x_{j-}\omega_j)$ if $j \in J$, and $\mu^t_j := l^t_j / (x_{j-}\omega_j)$. Now consider

$$\mu^t := \min\left\{1, \min_{j \in J \cup H} \mu^t_j\right\}$$

Define

$$c^t := \mu^t x + \left(1 - \mu^t\right)\omega.$$

First of all, $\mu^t \in [0, 1]$, which implies $c^t \in X$ due to convexity of X. Furthermore,

$$L^t \geq c^t - \omega \geq l^t$$

which follows from

$$\begin{aligned} l_j^t &\leq 0 \leq c_j^t - \omega_j = \mu^t (x_j - \omega_j) \leq \mu_j^t (x_j - \omega_j) = L_j^t \text{ for all } j \in J \\ L_j^t &\geq 0 \geq c_j^t - \omega_j = \mu^t (x_j - \omega_j) \geq \mu_j^t (x_j - \omega_j) = l_j^t \text{ for all } j \in H \\ L_j^t &\geq 0 = c_j^t - \omega_j \geq l_j^t \text{ for all } j \in K. \end{aligned}$$

Note that $\mu^t \to 1$ since

$$\begin{aligned} \mu_j^t &= \frac{L_j^t}{x_{j-}\omega_j} \to \frac{L_j}{x_{j-}\omega_j} \geq 1 \text{ for all } j \in J \\ \mu_j^t &= \frac{l_j^t}{x_{j-}\omega_j} \to \frac{l_j}{x_{j-}\omega_j} \geq 1 \text{ for all } j \in H \end{aligned}$$

But $\mu^t \to 1$ implies $c^t \to x$. Thus, $p^t (c^t - \omega) \to p (x - \omega)$. Therefore, $p (x - \omega) < \pi$ guarantees there is t' such that $p^t (c^t - \omega) \leq \pi^t$ for any $t > t'$. For $t \leq t'$, replace c^t with ω. This proves there is $c^t \in \beta^t$ such that $c^t \to x$ assuming $p (x - \omega) < \pi$.[2]

In case $p (x - \omega) = \pi$ define $\mathcal{L} := \{j \in H : l_j < 0\}$. If $\mathcal{L}$ is empty then $x - \omega \geq 0$. Suppose not. Then $\mathcal{L}$ is empty but there is j^* such that $x_{j^*} - \omega_{j^*} < 0$, but $l_{j^*} = 0$. However, $l_{j^*} \leq x_{j^*} - \omega_{j^*}$ holds by assumption. Contradiction. Hence, $\mathcal{L} = \emptyset$ and $\pi = 0$ implies $x - \omega = 0$ for $p^t \geq \underline{p} \in \mathbb{R}^n_{++}$. In this case, simply choose $x^t = x$ for each t, which has all desired properties. If $\mathcal{L} = \emptyset$ and $\pi > 0$ then define

$$\Lambda^t := \left\{ \lambda \in [0, 1] : \lambda p^t (x - \omega) \leq \pi^t,\ L^t \geq \lambda (x - \omega) \right\}.$$

Note that Λ^t is non-empty since $0 \in \Lambda^t$, and compact at each t. Therefore,

$$\lambda^t := \max_{\lambda} \Lambda^t$$

exists at each t. Now we shall prove that $\lambda^t \to 1$. To see that, suppose there is a t such that

$$\frac{\pi^t}{p^t (x - \omega)} > \lambda^t$$

and

$$L^t > \lambda^t (x_h - \omega_h) \text{ for all } h \in J$$

[2]Note that the proof up to this point demonstrates that if X is convex, and $0 \in X$ then $\{x \in X | L \geq x \geq l\}$ is well-defined and lower hemi-continuous in (L, l) for all $(L, l) \in \mathbb{R}^n_+ \times \mathbb{R}^n_-$. This observation will be used to prove the continuity of production plans with respect to quantity constraints.

and $1 > \lambda^t$. Then there would exist $\lambda \in \Lambda^t$ such that $\lambda > \lambda^t$. Contradiction. Therefore, at each t, $\lambda^t < 1$ implies

$$\lambda^t = \frac{\pi^t}{p^t(x-\omega)} \text{ or } L^t = \lambda^t(x_h - \omega_h) \text{ for some } h \in J$$

holds. However,

$$\frac{\pi^t}{p^t(x-\omega)} \to \frac{\pi}{p(x-\omega)} = 1$$

and

$$\frac{L^t}{x_h - \omega_h} \to \frac{L}{x_h - \omega_h} \geq 1.$$

These limits show that $\lambda^t \to 1$, which implies $\lambda^t x + (1-\lambda^t)\omega \to x$. Clearly, $\lambda^t x + (1-\lambda^t)\omega \in \beta^t$ at each t since $\lambda^t \in [0,1]$, which is the desired result for $\mathcal{L} = \emptyset$ and $\pi > 0$.

Now consider the case in which $\mathcal{L} \neq \emptyset$. Then define

$$\epsilon^t = \max\left\{\max_{j\in\mathcal{L}} l_j^t, \max_{j\in\mathcal{L}} \underline{x}_j - \omega_j\right\}$$

while the limit of ϵ^t is ϵ, that is, $\epsilon^t \to \epsilon$. Also define $\underline{\omega}^t$ by setting

$$\begin{aligned} \underline{\omega}_j^t &: = \omega_j + \epsilon^t \text{ for all } j \in \mathcal{L} \\ \underline{\omega}_j^t &: = \omega_j \text{ for all } j \notin \mathcal{L}. \end{aligned}$$

while the limit of $\underline{\omega}^t$ is $\underline{\omega}$, that is, $\underline{\omega}^t \to \underline{\omega}$. First of all

$$L^t \geq \underline{\omega}^t - \omega \geq l^t$$

by construction. Furthermore, $p(\underline{\omega}^t - \omega) < 0 \leq \pi^t$ at each t, and $\underline{\omega}^t \in X$ since $\underline{x} \leq \underline{\omega}^t \leq \omega$ (Assumption 2). In a similar vein, $L \geq \underline{\omega} - \omega \geq l$, $\underline{\omega} \in X$, and $p(\underline{\omega} - \omega) < 0$.

Now consider the sequence c^t, $t \in \mathbb{N}_+$, which is defined above. Note that, without loss of generality, we can assume

$$c^t \in X,\ L^t \geq c^t - \omega \geq l^t, \text{ and } c^t \to x.$$

If $p^t(c^t - \omega) > \pi^t$ then define

$$\lambda^t := \frac{p^t\omega + \pi^t - p^t\underline{\omega}^t}{p^t c^t - p^t\underline{\omega}^t} < 1.$$

If $p^t(c^t - \omega) \leq \pi^t$ then let $\lambda^t = 1$. Now define

$$d^t := \lambda^t c^t + (1-\lambda^t)\underline{\omega}^t \in X.$$

One can show that

$$\begin{aligned} d^t - \omega &= \lambda^t \left(c^t - \omega\right) + \left(1 - \lambda^t\right) \left(\underline{\omega}^t - \omega\right) \geq l^t \\ d^t - \omega &= \lambda^t \left(c^t - \omega\right) + \left(1 - \lambda^t\right) \left(\underline{\omega}^t - \omega\right) \leq L^t. \end{aligned}$$

Moreover, $p^t \left(c^t - \omega\right) > \pi^t$ implies

$$p^t d^t = p^t c^t \frac{p^t \omega + \pi^t - p^t \underline{\omega}^t}{p^t c^t - p^t \underline{\omega}^t} + \frac{p^t c^t - p^t \omega - \pi^t}{p^t c^t - p^t \underline{\omega}^t} p^t \underline{\omega}^t = p^t \omega + \pi^t$$

On the other hand, $p^t \left(c^t - \omega\right) \leq \pi^t$ implies $p^t d^t = p^t c^t \leq p^t \omega + \pi^t$. That proves $d^t \in \beta^t$. Now we shall show that $d^t \to x$. Since $c^t \to x$

$$\frac{p^t \omega + \pi^t - p^t \underline{\omega}^t}{p^t c^t - p^t \underline{\omega}^t} \to \frac{p\omega + \pi - p\underline{\omega}}{px - p\underline{\omega}} = \frac{px - p\underline{\omega}}{px - p\underline{\omega}} = 1$$

which ensures that $\lambda^t \to 1$. Therefore, $d^t \to x$. That completes the proof for lower hemi-continuity since $d^t \in \beta^t$.

Regarding upper hemi-continuity, note that β^t is convex, compact, and l.h.c. Thus β^t is also u.h.c. due to Lemma 1 in Hildenbrand (1974, p.33). Consequently, β^t is continuous. □

As a corollary, continuity of the set of production plans with respect to quantity constraints ensues.

Corollary 5. *Assume $0 \in Y$, and Y is convex.Then the correspondence $\Gamma : \mathbb{R}^n_+ \times \mathbb{R}^n_- \to \mathbb{R}^n$ given by*

$$\Gamma\left(L, l\right) = \left\{y \in Y : L \geq y \geq l\right\}$$

is continuous in $\left(L, l\right)$.

Proof. Consider

$$\beta\left(p, L, l, \pi\right) = \left\{x \in X : p\left(x - \omega\right) \leq \pi,\ L \geq x - \omega \geq l\right\}$$

as defined in Theorem 4, and impose $\omega = 0$ and $X = Y$. Pick any $\left(p, L, l, \pi\right) \in P \times \mathbb{R}^n_+ \times \mathbb{R}^n_- \times \mathbb{R}_+$ such that $pL < \pi$, which implies $p\left(x - \omega\right) \leq \pi$ is a slack constraint. Thus the first part of Theorem 4, which deals with the case of $pL < \pi$, applies for the lower hemicontinuity of $\beta\left(p, \cdot, \cdot, \pi\right)$. However, $pL < \pi$ implies $\beta\left(p, L, l, \pi\right) = \Gamma\left(L, l\right)$ since $\omega = 0$ and $X = Y$. This proves the lower hemi-continuity of $\Gamma\left(\cdot, \cdot\right)$ (see also Footnote 2).

As for the upper hemicontinuity, Lemma 1 in Hildenbrand (1974, p.33) again applies. This completes the proof. □

Now we use these two results to prove the existence of Drèze equilibrium:

Theorem 6. *For any $\mathcal{E}$, which satisfies Assumption 1-5, there is a Drèze equilibrium.*

Proof. Define $(x_1, ..., x_r, y_{r+1}, ..., y_m) \in A \subset \mathbb{R}^{nm}$ if and only if $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j and $\sum x_i - \sum \omega_i - \sum y_j = 0$. Then define the set $\widehat{X}_i$ of attainable consumption plans for consumer i, i.e., the projection of A on X_i. Similarly, we define the set $\widehat{Y}_j$ of attainable production plans for producer j.

The assumptions on the consumption sets and on the production sets ensure that A is a compact set. Let K be a closed cube of $\mathbb{R}^n$ with center 0 and containing in its interior the $\widehat{X}_i$'s and the $\widehat{Y}_j$'s. Let $2a$ be the length of one of its sides. Then define $\overline{X}_i := X_i \cap K$ and $\overline{Y}_j := Y_j \cap K$. These sets are compact.

Now consider the compact, convex set $Q := \{q \in \mathbb{R}^n | a \geq q_i \geq -a, \, i = 1, ..., n\}$.

For every $q \in Q$ define the rationing scheme $(p(q), L(q), l(q)) \in P \times \mathbb{R}^n_+ \times \mathbb{R}^n_-$ by

$$\begin{aligned} p_h(q) \ &: \ = \min\left\{\overline{p}_h, \max\left\{q_h, \underline{p}_h\right\}\right\} \\ L_h(q) \ &: \ = \overline{p}_h + a - \max\{\overline{p}_h, q_h\} \\ l_h(q) \ &: \ = \underline{p}_h - a - \min\left\{\underline{p}_h, q_h\right\} \text{ s.t. } h = 1, ..., n. \end{aligned}$$

Let $Q' := \{q \in Q | q_1 = 1\}$. Obviously, $(L(q), l(q))$ is a continuous vector-valued function of q, for all $q \in Q'$.

Since Y_j is closed, convex, non-empty subset of $\mathbb{R}^n$ the correspondence defined by $\left\{y_j \in \overline{Y}_j | L \geq y \geq l\right\}$ is continuous due to Corollary 5. hence, by continuity of $(L(\cdot), l(\cdot))$, the correspondence defined by $\left\{y_j \in \overline{Y}_j | L(q) \geq y \geq l(q)\right\}$ is also continuous. Thus $\eta_j(q)$, which is defined as profit maximizing production plans on Q', is u.h.c., and $\pi_j(q) := p(q)\eta_j(q)$ is continuous. Furthermore, $\eta_j(q)$ is non-empty and convex for all q.

For the sake of notational clarity define $\gamma_i(q) := \beta_i(p(q), L(q), l(q))$. Let $\xi_i(q)$ be the subset of $\gamma_i(q)$ consisting of those elements which are maximal for $u_i : X_i \rightarrow \mathbb{R}$. Due to continuity of $\pi_j(q)$, $\gamma_i(q)$ is a continuous correspondence from Q' to X_i. Therefore the correspondence $\xi_i(q)$, from Q' to X_i, is non-empty and convex and u.h.c because $\gamma_i(q)$ is compact, continuous on Q' and u_i is continuous and quasi-concave.

Let us now consider the correspondence μ defined on $Z := \prod \overline{X}_i \times \prod \overline{Y}_j$ by

$$\begin{aligned} & \mu(x_1, ..., x_r, y_{r+1}, ..., y_m) \\ = \ & \left\{q' \in Q' | q' \text{ maximizes } q\left(\textstyle\sum_i x_i - \sum_i \omega_i - \sum_j y_j\right) \text{ on } Q'\right\} \end{aligned}$$

Consider finally the correspondence $\varphi := \mu \times \prod \xi_i \times \prod \eta_j$ from $Z \times Q'$ into itself. It is u.h.c and non-empty and convex valued. Furthermore, $Z \times Q'$ is non-empty, convex and compact. Thus by Kakutani's fixed point theorem, the correspondence φ has a fixed point. Let us denote by $\left(x_1^*, ..., x_r^*, y_{r+1}^*, ..., y_m^*, q^*\right)$ such a fixed point, i.e., $x_i^* \in \xi_i\left(q^*\right)$ for all i, $y_j^* \in \eta_j\left(q^*\right)$ for all j, $q^* \in Q'$, $\left(q^* - p\right) z \geq \left(q - p\right) z$ for all $q \in Q'$, where $z = \sum_i x_i^* - \sum_i \omega_i - \sum y_j^*$.

If there is h such that $z_h > 0$ then $q^* = a$, which implies $L_h\left(q^*\right) = 0 \geq x_i^* - \omega_i$ and $L_h\left(q^*\right) = 0 \leq y_j^*$ for all i and j. This yields $z_h \leq 0$. Contradiction. The case for $z_h < 0$ is similar. Thus, $z = 0$, that is $x_i^* \in \widehat{X}_i$ for all i and $y_j^* \in \widehat{Y}_j$. These are sufficient to conclude conditions (i)-(ii) and (iii) are satisfied. As for (iv) note that $L_h\left(q^*\right) < a$ implies $l_h\left(q^*\right) = -a$ and $l_h\left(q^*\right) > -a$ implies $L_h\left(q^*\right) = a$. Regarding (v) note that $\overline{p}_h > p_h\left(q^*\right)$ implies $l_h\left(q^*\right) = -a$, and $\underline{p}_h < p_h\left(q^*\right)$ implies $L_h\left(q^*\right) = a$. These show conditions (iv-v) are also satisfied, and complete the proof. □

CHAPTER 4

On Voluntary and Efficient Allocations

4.1. Introduction

An allocation is said to be voluntary if no agent can benefit by trading less at the prevailing prices, (i.e. no forced trade.). A voluntary allocation is strongly voluntary if there is at least one good (e.g. money) with a clear market.

Hence, voluntariness is germane to how markets operate, and often deemed a defining tenet of market games in economic theory (see Bevia et al (2003), Svensson (1991), Benassy (1986), Drèze (1975)). Voluntariness also proved to be a crucial condition regarding efficiency and competitiveness of allocation mechanisms. In particular, Silvestre (1985) shows that, under certain hypotheses, an allocation is voluntary and efficient if, and only if, it is Walrasian. This essay presents a generalization of this equivalence.

Our interest in generalizing Silvestre (1985) stems from widespread and persistent non-clearing market phenomenon, which constitutes the rule rather than the exception in real market economies. To the best of our knowledge, Silvestre's equivalence provides the most general conditions to test Pareto-efficiency of non-clearing markets. More specifically, this result implies that voluntary trade in non-clearing markets is inefficient.

However, albeit the widespread conviction amidst economists, the inefficiency of disequilibrium is not unconditional. The assumptions for this important result of Silvestre are convexity, smoothness, zero profits, and interior consumption, that is, strictly positive consumption of all goods by all individuals. For the cases in which profits are positive, the result still holds if voluntariness is replaced by its mildly stronger version –strong voluntariness.

Since disequilibrium is ubiquitous in all economies these results seem to suggest that real markets exhibit persistent inefficiencies. We find this conclusion very important. However, interior consumption condition is a heroic proviso, and impedes applying Silvestre's results on real world disequilibrium phenomenon. First of all, Silvestre concocts a robust example where his results fail when interior consumption condition is violated. We also present an example in this fashion. Hence interior consumption condition is not an innocuous assumption from a theoretical point of view. However, there is no general result to test the efficiency of disequilibrium when consumption is on the boundary.

Nevertheless, consuming all goods in strictly positive amounts (i.e. interior consumption) for any single consumer is clearly impossible. For one thing, this would require simultaneous consumption at two distinct places when goods are distinguished over time and space. Furthermore, that an individual can never consume someone else's time as her leisure induces pervasive boundary consumption – as opposed to interior consumption – in the space of all labor types.

In general, as Moore (2007) lucidly expresses, "if we define commodities finely ... all reasonable allocations would result in each consumer's commodity bundle being on the boundary of its consumption set" and "consequently, even though we tend to think of boundary values as being a very special case, they are the norm in reality. (p. 222)" In a similar fashion, Geanakoplos (2004) states that "the more finely the commodities are described, the less likely are the commodity markets to have many buyers and sellers (p.116)." Arrow (1951) also notes that:

> "Indeed, for any one individual, it is quite likely that the number of commodities on the market of which he consumes nothing exceed the number which he uses in some degree (p. 509)."

Furthermore, boundary consumption is likely to be more pertinent over time since the gamut of products is expanding constantly. On the other hand, as we stated above, there is no welfare analysis of disequilibrium, which is the norm in all modern economies, without interior consumption condition.

Silvestre (p. 813) concedes the unrealistic nature of interior consumption, and conjectures that it may be relaxed to a condition that we refer to as quasi interiority. Quasi interiority is a compound of two conditions. According to the first one "each individual consumes some amount of what she owns initially". For example, workers rest for some time. The second one prescribes that "each individual consumes something that is also valuable to others".

This study verifies Silvestre's conjecture when profits are zero. That is to say, we show that voluntary, efficient, and quasi interior allocations are Walrasian when profits are zero in any convex smooth economy.

The second result is concerned with positive profit cases. We show that any strongly voluntary, efficient and weakly interior allocation is a Walrasian equilibrium for smooth convex economies. Note that in this case, we invoke weak interiority, which is defined as follows. Consider a quasi interior allocation such that there is a clear market of a particular good, say money. If on both sides (short vs. long) of each market there is a consumer who trades money then the allocation is said to be weakly interior.

As far as we know, these two findings present the weakest conditions in the literature for non-Walrasian allocations to be Pareto-inefficient. The concept of quasi interiority may also contribute to incomplete markets literature, in which interior consumption is very common (e.g. Magill and Quinzii (2002)).

4.2. Definitions and notation

A private ownership economy $\mathcal{E}$ is comprised of a set of commodities $N = \{1, ..., n\}$, a set of consumers $C = \{1, ..., s\}$, and a set of firms $F = \{s+1, ..., m\}$. Each consumer $a \in C$ is characterized by a consumption set $X_a = \mathbb{R}^n_+$, a utility function u_a defined on X_a, an initial vector of commodities $\omega_a \in \mathbb{R}^n_+$, and a vector of profit shares $\theta^a \in \mathbb{R}^{m-s}_+ \left(\sum_{a \in C} \theta^a = 1\right)$. Utility functions are concave, nonsatiated, nondecreasing in every argument, and smooth on $\mathbb{R}^n_{++}$. Each firm $a \in F$ has a convex production set $Y_a \subset \mathbb{R}^n$ such that $Y_a \cap \mathbb{R}^n_+ = \{0\}$.

In the analysis that follows, we stipulate that the production technology is smooth.

Assumption 13. *$\forall \overline{y} \in Y_a$ such that $\overline{y}_j < 0$ and $\overline{y}_h > 0$ there exists $\epsilon > 0$ and a differentiable function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi\left(\overline{y}_j\right) = \overline{y}_h$ and, $y_j \in \left(\overline{y}_j - \epsilon, \overline{y}_j + \epsilon\right)$ implies*

$$\left(\overline{y}_j, ..., \overline{y}_{j-1}, y_j, \overline{y}_{j+1}, ..., \overline{y}_{h-1}, \psi\left(y_h\right), \overline{y}_{h+1}, ..., \overline{y}_n\right) \in Y_a.$$

Hereafter $\mathcal{E}$ is assumed to satisfy all the assumptions stated up to this point. Next we define the concept of feasibility.

Definition 10. *An allocation $w^* = \left(x_1^*, ..., x_s^*, y_{s+1}^*, ..., y_m^*\right) \in \mathbb{R}^{nm}$ is an $m-$tuple of s consumption vectors and $m-s$ production vectors; w^* is feasible iff $x_a^* \geq 0$, $\forall a \in C$, $y_a^* \in Y_a$,$\forall a \in F$, and: $\sum_{a \in C} x_a^* - \omega_a \leq \sum_{a \in F} y_a^*$.*

One of the central concepts in this study is voluntariness, which is defined as follows:

Definition 11. *A feasible allocation w^* is voluntary for $(p, \mathcal{E})$, $p = (p_1, ..., p_n)$, iff: (i) $\forall a \in C$, maximizes $u_a(x_a)$ subject to: $p(x_a - \omega_a) = \sum_{b \in F} \theta_b^a p y_b^*$, and to:*

$$\min\left\{0, x_{aj}^* - \omega_{aj}\right\} \leq x_{aj} - \omega_{aj} \leq \max\left\{0, x_{aj}^* - \omega_{aj}\right\}$$

(ii) $\forall a \in F$, y_a^* maximizes $p y_a$ subject to: $y_a \in Y$, and to:

$$\min\left\{0, y_{aj}^*\right\} \leq y_{aj} \leq \max\left\{0, y_{aj}^*\right\}$$

where $j = 1, ..., n$.

Voluntariness requires that, at the given prices, trading less is detrimental for any agent. The next definition delineates strongly voluntary allocations.

Definition 12. *A feasible allocation w^* is strongly voluntary for $(p, \mathcal{E})$ iff (i) w^*satisfies condition (ii) of Definition 3, and (ii) $\forall a \in C$, maximizes $u_a(x_a)$ subject to: $p(x_a - \omega_a) = \sum_{b \in F} \theta_b^a p y_b^*$, and to:*

$$\min\left\{0, x_{aj}^* - \omega_{aj}\right\} \leq x_{aj} - \omega_{aj} \leq \max\left\{0, x_{aj}^* - \omega_{aj}\right\}$$

where $j = 2, ..., n$.

The key difference between voluntary and its stronger counterpart is that there is at least one market in which no consumer $a \in C$ is quantity constrained at a strongly voluntray allocation. This good is generally interpreted as the medium of exchange (e.g. money) in the standard disequilibrium literature.

For the sake of completeness, the definitions of Walrasian equilibrium and Pareto-efficient allocations are also given:

Definition 13. *A feasible allocation w^* is a Walrasian equilibrium for the private ownership economy $\mathcal{E}$ and for the price vector p iff (i)*

$$p_j\left(\sum_{a \in C} x_{aj}^* \leq \sum_{a \in C} \omega_{aj} + \sum_{a \in F} y_{aj}^*\right) = 0,$$

$\forall j \in N$; (ii) x_a^* maximizes $u_a(x_a)$ subject to:

$$p(x_a - \omega_a) \leq \sum_{b \in F} \theta_b^a p y_b^*,$$

$\forall a \in C$; (iii) y_a^* maximizes $p y_a$ subject to $y_a \in Y_a$, $\forall a \in F$.

Definition 14. *A feasible allocation w^* is Pareto efficient iff there does not exist another feasible allocation $\overline{w}$ such that $u_a(\overline{x}_a) \geq u_a(x_a^*)$, $\forall a \in C$, with strict inequality for at least one $a \in C$.*

In lieu of interior consumption, (i.e. $x_a^* \gg 0$, $\forall a \in C$) which is common in disequilibrium welfare analysis, we invoke a weaker a assumption, to wit, quasi interiority which is defined as follows:

Definition 15. *A feasible allocation w^* is quasi interior iff $\forall a \in C$ (i) $\omega_{aj} > 0$ implies $x_{aj}^* > 0$, $\forall j \in N$; and (ii) there is a feasible allocation w' such that $u_b(x_b') \leq u_b(x_b^*)$ implies $a = b$.*

The first condition says that no one supplies up to her full capacity, (e.g. workers rest for some time). The second condition entails that each individual consumes something valuable (of course, in utility terms) for the rest of the society.[1] It is an intriguing fact that this is actually related to the existence of Walrasian equilibrium (see Nayak (1982) and Silvestre (1985)). However, in the parlance of Maskin and Tirole (1984, p.323), the interpreation of the second condition is that "all traders have strictly positive weights" in the standard linear welfare program.

Now we can state our first result:

Theorem 7. *Let $p^* \gg 0$ and w^* be a Pareto-efficient, voluntary, and quasi interior allocation for $(p^*, \mathcal{E})$ with $p^* y_a^* = 0$, $\forall a \in F$ (zero-profits). Then there exists a $\widehat{p} \in \mathbb{R}_+^n$ such that w^* is a Walrasian equilibrium for $(\widehat{p}, \mathcal{E})$.*

Proof. See Appendix A-2. □

Remark 3. *This theorem proves the conjecture of Silvestre (1985, p.813)*

Remark 4. *Proposition 3 of Maskin and Tirole (1984, p.323), Theorem 1 of Silvestre (1985), Proposition 3.3 of Herings and Konovalov (2008) are corollaries of this theorem.*

The next step is to cover the cases in which profits are positive. To this end, I stipulate strong voluntariness, and weak interiority, where the latter is defined as:

Definition 16. *A feasible allocation w^* is weakly interior iff (i) w^* is quasi interior, and (ii) $\forall (a, j) \in C \times N$, $x_{aj}^* - \omega_{aj} \neq 0$ implies*

$$x_{b1}^* \left(x_{bj}^* - \omega_{bj}\right)\left(x_{aj}^* - \omega_{aj}\right) > 0$$

for some $b \in C$.

According to weak interiority, on both sides (short vs. long) of each market there is a consumer, who is trading good 1. Interpreting good 1 as the medium of exchange such as money can substantiate its particular role. A simple but coarse way to pass from quasi interiority to weak interiority is to assume that everyone consumes some amount of good 1. It is noteworthy to state that even this coarse version of weak interiority is weaker than interior consumption (i.e. $x_a^* \gg 0$, $\forall a \in C$).

When profits are positive, our second result ensues:

Theorem 8. *Let $p^* \gg 0$ and w^* be a Pareto-efficient, strongly voluntary, and weakly interior allocation for $(p^*, \mathcal{E})$. Then there exists a $\widehat{p} \in \mathbb{R}_+^n$ such that w^* is a Walrasian equilibrium for $(\widehat{p}, \mathcal{E})$.*

[1] This condition can readily be relaxed to the following weaker but more complex form: $\forall (a, b, j) \in C \times C \times N$ write $a\,[j]\,b$ iff $x_{aj}^* > 0$ and $\partial u_b(x_b^*)/\partial x_{bj}^* > 0$. Then $\forall (a_0, a_t) \in C \times C$ there exists a chain of $(a_1, ..., a_{t-1}) \in C^{t-1}$ and $(j_1, ..., , j_t) \in N^t$ such that $a_i\,[j_{i+1}]\,a_{i+1}$, $i = 0, ..., t-1$.

Proof. See Appendix B-2. □

Remark 5. *Theorem 2 of Silvestre (1985) is a corollary of this theorem.*

Now an example will be presented. It aims at demonstrating that weak interiority is indispendisble when profits are non-zero. This example also shows that the validity of Silvestre's conjecture is restricted to allocations with zero profits.

Example 9. *If profits are positive then strongly voluntary, Pareto efficient, and quasi interior allocations may not be Walrasian (i.e. weak interiority is indispensible). Consider two consumers, one firm, and three goods. The price vector is* $p^* = (2, 3, 2)$*. The production technology is*

$$Y = \left\{ y \in \mathbb{R}^3 | y_2 \leq 0, y_3 \leq 0, y_1 \leq \frac{5}{2} (-y_3)^{\frac{2}{5}} \right\} + \mathbb{R}^3_-$$

while the production plan is $y^* = \left\{ \frac{5}{2}, 0, -1 \right\}$*. The consumers' initial endowments, consumption plans, profit shares, and marginal utilities are given by*

$$\begin{aligned} \omega_1 &= (0,1,0) \text{ and } \omega_2 = (0,2,2) \\ x_1^* &= (0,2,0) \text{ and } x_2^* = \left(\frac{5}{2}, 1, 1 \right) \\ \theta^1 &= 1 \text{ and } \theta^2 = 0 \\ du_1^* &= (1,2,1) \text{ and } du_2^* = (2,2,2) \end{aligned}$$

where $du_a^* := (\partial u_a / \partial x_{a1}, \partial u_a / \partial x_{a2}, \partial u_a / \partial x_{a3})$*, evaluated at* x_a^**,* $a = 1, 2$*.*
First, let us see that w^* *is Pareto-efficient. Keeping the production fixed, the only way to maintain Pareto-improvement would be that individual* 1 *consumes less of either good* 1 *or good* 3*.*[2] *But this is not feasible since individual* 1 *does not consume any of good* 1 *or* 3*. Pareto-improvement through the conduit of altering the production plan requires the firm to change its production plan regarding good* 1 *or good* 3*. But this change would not bring about Pareto-improvement since the firm's marginal rate of transformation and individual* 2*'s marginal rate of substitution between good* 1 *and good* 3 *are equal to each other. Moreover* w^* *is strongly voluntary since all agents' decisions are optimal under the constraint that individual* 2*'s supply of good* 3 *is constrained. Quasi interiority of* w^* *is obvious.*
However, w^* *cannot be a competitive equilibrium for equilibrium requires proportionality between* du_2^* *and* p^**.*

[2]The difference in individual 1's consumption would be transferred to individual 2, who would compensate individual 1's loss of utility by transferring some good 2 to individual 1.

CHAPTER 5

Conclusion

We conclude with discussing the major weaknesses of this thesis.

First of all, we invoked a rather stringent definition of being a worker when the nonexistence of competitive equilibrium is demonstrated in Chapter 2. In particular, we say that a worker is a consumer whose income is solely derived from supplying labor.

As we discussed earlier, this definition does not rule out enjoying capital income through savings. Yet we conjecture that all our results continue to hold with a substantially weaker form of being a worker. More specifically, admitting workers to possess a sufficiently small amount of standard goods – commodities subject to satiation – as initial endowments would not affect the results. Moreover, owning sufficiently small profit shares of the firms that produce only standard goods could be appended to our model as well.

Consequently, our results do not cover economies up to the full generality that is possible since our definition of worker could be weaker.

Another negative feature of Chapter 2 is germane to the redistribution of income as a policy implication. Although we show that there always exists a redistribution scheme among consumers so that a competitive equilibrium exists, we do not provide any details in this regard. For instance, our results do not tell the minimum amount of redistribution that supports a competitive equilibrium.

Secondly, we used convexity heavily all through the thesis without exception. Nonetheless, convexity is a very strong assumption for production sets. Indeed, fixed costs – a basic notion of modern technology – cause substantial non-convexities.

In the case of non-convexities, neither the existence of non-Walrasian equilibrium nor the inefficiency properties of disequilibrium ensue – subjects that are discussed in Chapter 3 and 4. As a matter of fact, we believe that Pareto-efficient disequilibrium due to non-convexities in production constitute a very interesting subject for future research.

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APPENDIX

A-1

Proof. Fix a vector $(\alpha, \beta, a, b, m_1, m_2, K) \in \mathbb{R}^7_{++}$. Write $\mathcal{E}^t := (\alpha, \beta, a, b, m_1, m_2, K, t)$ where t is the number of workers. Now consider a sequence of economies $\mathcal{E}^t$, $t = 1, 2, ...$ Competitive equilibrium of $\mathcal{E}^t$ is denoted by

$$\left(y^t, x_1^t, ..., x_{K+t}^t, p^t\right) \in Y \times X_1 \times ... \times X_{K+t} \times \mathbb{R}^5_+.$$

where $p^t = (1, p_2^t, p_3^t, p_4^t, p_5^t)$ is the competitive equilibrium price vector at the t^{th} step. So the price of gold is the numeraire.

Suppose the claim is not true. That is, for all t there exists $r > t$ such that $\left(y^r, x_1^r, ..., x_{K+t}^r, p^r\right)$ is nonempty. Now collect all such r's to construct a subsequence such that $\left(y^r, x_1^r, ..., x_{K+t}^r, p^r\right)$ is nonempty at each r. Now we will maintain a contradiction:

Step 1: Neither p_2^r nor p_3^r can be zero at any r. Were $p_3^r = 0$ at some r then no one would supply labor. If there is no labor supply then production of gold and bread should also be zero. That is to say, $y_1^r = y_2^r = y_3^r = 0$. Consequently, the optimal consumption of all individuals should be zero as well. However, the capitalists consume zero only if their income is zero. So $p_4^r = p_5^r = 0$ must be the case in order to ensure that the incomes of the capitalists are zero. In this case, the profit maximization is not well-defined due to constant returns to scale technology for all inputs are free but the price of gold is 1. Contradiction with optimization.

Were $p_2^r = 0$ for some r then bread would be a free good. Thus all consumers would demand bread but the firm would not produce any for $p_3^r > 0$ due to Step 1. Thus, $p_2^r > 0$ for all r.

Step 2: Given p^r, each worker i solves

$$\max_{x_i} \sum_{h=1,2,3} x_{ih}^{\alpha}$$

subject to

$$p_3^r \geq p^r x_i \text{ and } x_i \in \mathbb{R}_+ \times [0, \beta] \times [0, 1] \times \mathbb{R}^2_+.$$

The standard computations show that the solution $x_i^r = (x_{i1}^r, ..., x_{i5}^r)$ satisfies the following property:

$$\text{if } x_{i2}^r < \beta \text{ then } x_{i3}^r = \frac{1}{1 + \left(\frac{p_3^r}{p_2^r}\right)^\eta + (p_3^r)^\eta} \tag{.1}$$

where $\eta = \alpha / (1 - \alpha)$.

Step 3: In this step, we shall prove that $x_{r3}^r \to 1$ as $r \to \infty$. Given p^r, the firm solves

$$\max_{y \in Y} p^r y$$

Let us first see that if y^r is a solution to this optimization problem then $y_1^r + y_2^r + y_3^r = 0$. Were $y_1^r + y_2^r + y_3^r < 0$ then there would be room to decrease labor demand without altering output. However, this implies that y^r cannot be profit maximizing since $p_3^r > 0$ in equilibrium due to Step 1. We deduce $y_1^r + y_2^r + y_3^r = 0$. But

$$ay_1 + y_4 \leq 0 \text{ and } by_2 + y_5 \leq 0$$

give

$$ay_1^r \leq Km_1 \text{ and } by_2^r \leq Km_2$$

since $y_4 + Km_1 \geq 0$ and $y_5 + Km_2 \geq 0$. That is, bread and gold production are curbed by the existing stock of machinery. Juxtaposing $ay_1^r \leq Km_1$ and $ay_2^r \leq Km_2$ and $y_1^r + y_2^r + y_3^r = 0$ implies

$$y_3^r + \lambda \geq 0 \text{ where } \lambda := K\left(\frac{m_1}{a} + \frac{m_2}{b}\right)$$

The interpretation of this result is that λ is an upper bound of labor demand by the firm. But, due to labor market clearing,

$$\sum_{i \in W} (1 - x_{i3}^r) \leq -y_3^r \leq \lambda$$

where $1 - x_{i3}^r$ is workers i's labor supply. So $x_{r3}^r \to 1$ as $r \to \infty$ for all i.

Step 4: Now we will state a self-evident fact, which ensues due to profit maximization with a fixed coefficient production technology:

$$p_2 - p_3 - bp_5 \leq 0 \text{ (with equality if } y_2 > 0).$$

Step 5: The sum of all workers' demand for good 2 (i.e. bread) is $\sum_{i \in W} x_{i2}^r$. Therefore, due to feasibility,

$$\sum_{i \in W} x_{i2}^r \leq K\frac{m_2}{b}.$$

This shows that $x_{r2}^r \to 0$ as $r \to \infty$. However, due to Eq. .1, $x_{r2}^r \to 0$ and $x_{r3}^r \to 1$ imply $\frac{p_3^r}{p_2^r} \to 0$. Thus, applying Step 4, there is r_1 such that $r > r_1$ implies $p_5^r > 0$. But if $p_5^r > 0$ then the supply of bakery machinery is Km_2. This assures that $y_5^r + Km_2 = 0$ due to market clearing. Therefore, $y_2^r = Km_2/b$.

Step 6: Now we shall prove that

$$\sum_{i \in W} x_{i2}^r \leq \frac{p_3^r}{p_2^r} \lambda.$$

Note that

$$x_{i2}^r \leq \frac{p_3^r}{p_2^r} (1 - x_{i3}^r) \tag{.2}$$

holds due to budget constraint of the workers, and $\sum_{i \in W} (1 - x_{i3}^r) \leq \lambda$ due to Step 3. Therefore,

$$\sum_{i \in W} x_{i2}^r \leq \frac{p_3^r}{p_2^r} \sum_{i \in W} (1 - x_{i3}^r) \leq \frac{p_3^r}{p_2^r} \lambda.$$

Step 7: Assume $r > r_1$ which ensures $y_2^r = Km_2/b$ due to Step 5. The capitalists' demand for bread is bounded from above by their satiation parameter β. Thus the market clearing condition for bread gives

$$\frac{Km_2}{b} - K\beta - \frac{p_3^r}{p_2^r} \lambda \leq y_2^r - \sum_{i=1,\ldots,r+K} x_{i2}^r = 0 \tag{.3}$$

for all r. On the other hand, there is r_2 such that $r > r_2$ yields

$$\frac{Km_2}{b} - K\beta - \frac{p_3^r}{p_2^r} \lambda > 0$$

since $\frac{p_3^r}{p_2^r} \to 0$ due to Step 5 and $m_2 - b\beta > 0$ by assumption. This inequality contradicts Eq. (.3), and competitive equilibrium cannot exist if $r > \max\{r_1, r_2\}$.
$\square$

APPENDIX

B-1

Suppose not. Then $p \gg 0$, $W > W^*$, and there is a Pareto-efficient and voluntary allocation ξ. Now we shall maintain a contradiction.

Step 1: This step proves $x_{i1} > 0$ for all i. Note that $pe_i + \theta_i py > 0$ for all i since $p \gg 0$. However, there is no constraint on good 1 according to the definition of voluntariness. Were $x_{i1} = 0$ for some i then individual i would not be maximizing utility.

Step 2: This step proves $x_{ih} > 0$ for all i and for all $h = 1, 2, 3$. By hypothesis ξ is Pareto-efficient. Therefore ξ solves the following linear welfare program:

$$\max_{\xi} \sum_i \lambda_i u_i \left(x_i \right) \text{ s.t. } \xi \text{ is feasible}$$

for some $\lambda \in \mathbb{R}_+^{W+K}$ such that $\sum_i \lambda_i > 0$.

But $x_{i1} > 0$ due to Step 1. This ensures that $\lambda_i > 0$ for all i. Suppose not. Then there would be an individual i with $\lambda_i = 0$ and another individual j with $\lambda_j > 0$. In this case there exists another allocation where individual i consumes less and individual j consumes more gold (i.e. good 1). Deduce that ξ cannot solve the linear welfare program.

Since $\lambda_i > 0$ for all i, it follows that $x_{ih} > 0$ for all i and for all $h = 1, 2, 3$. Had there been an individual i and a product h such that $x_{ih} = 0$ then $\partial u_i / \partial x_{ih} = \infty$.

Step 3: This step proves that ξ is a competitive equilibrium. Since $p \gg 0$, ξ is Pareto-efficient and voluntary, and $x_{ih} > 0$ for all i and for all $h = 1, 2, 3$, Silvestre (1985, Thereom 2) applies. That is, ξ is a competitive equilibrium.

However, $W > W^*$ by assumption, which implies that there is no competitive equilibrium. Contradiction.

APPENDIX

C-1

Proof. For any $\mathcal{E}$ the Second Fundamental Theorem of Welfare applies. Therefore, any Pareto-efficient allocation ξ can be supported as a competitive equilibrium with some lump-transfers q.

The rest of the proof is a generalization of the proof of Proposition 1. Fix a vector $(\alpha, \beta, a, b, m_1, m_2, K) \in \mathbb{R}^7_{++}$. Write $\mathcal{E}^t := (\alpha, \beta, a, b, m_1, m_2, K, t) \in \mathbb{R}^8_{++}$ where t is the number of workers. Now consider a sequence of economies $\mathcal{E}^t$, $t = 1, 2, ...$ Let

$$\xi^t := \left(y^t, x_1^t, ..., x_{K+t}^t\right) \in Y \times X_1 \times ... \times X_{K+t}.$$

be an arbitrary Pareto-efficient allocation. Write $p^t = (1, p_2^t, p_3^t, p_4^t, p_5^t)$ for any competitive equilibrium price vector and q^t for any lump-sum transfers that support the equilibrium at the t^{th} step.

Suppose the claim is not true. That is, for all t there exists $r > t$ such that

$$\sum_{i \in W} q_i^r \leq 0 \leq \sum_{i \in K} q_i^r$$

for some q^t. Now collect all such r's.

After that point, Step 1-7 of the proof of Proposition 1 apply after making the following changes. Replace Eq (.1) with

$$\text{if } x_{i2}^r < \beta \text{ then } x_{i3}^r = \frac{1 + \frac{q_i^r}{p_3^r}}{1 + \left(\frac{p_3^r}{p_2^r}\right)^\eta + (p_3^r)^\eta}$$

and Eq (.2) with

$$x_{i2}^r \leq \frac{p_3^r}{p_2^r} (1 - x_{i3}^r) + \frac{q_i^r}{p_2^r}.$$

But the conclusion is the nonexistence of competitive equilibrium if $r > \max\{r_1, r_2\}$. Contradiction with q^r supports an equilibrium at each r. □

APPENDIX

A-2

First we present the tangency conditions for voluntary, strongly voluntary, and Pareto-efficient allocations. Let w^* be a feasible allocation for $(p^*, \mathcal{E})$. Define $\partial u_a(x_a^*)/\partial x_{aj} := u_{aj}^*$. If w^* is a voluntary allocation then $\forall a \in C$,$\forall j \in N$, there is $\left(\alpha_a, \overline{\gamma}_{aj}, \underline{\gamma}_{aj}, \mu_{aj}\right) \in \mathbb{R}_+^4$ such that

$$u_{aj}^* - \alpha_a p_j^* - \overline{\gamma}_{aj} + \underline{\gamma}_{aj} + \mu_{aj} = 0$$

where

$$\begin{aligned}
\overline{\gamma}_{aj} &= 0 \text{ if } x_{aj}^* - \omega_a < 0 \\
\underline{\gamma}_{aj} &= 0 \text{ if } x_{aj}^* - \omega_a > 0 \\
\mu_{aj} &= 0 \text{ if } x_{aj}^* > 0.
\end{aligned}$$

for $j = 1, ..., n$.

If w^* is a strongly voluntary allocation then all above conditions remain the same but the following additional complementary slackness condition is imposed:

$$\overline{\gamma}_{a1} = \underline{\gamma}_{a1} = 0, \forall a \in C.$$

These first order conditions regarding consumer optimization are immediate results that ensue from Karush-Kuhn-Tucker Theorem.

On the other hand, any Pareto-efficient allocation w^* is optimal for a linear welfare program, where welfare weights are appropriately chosen. More specifically, if w^* is a Pareto-efficient allocation then there exists $\lambda_a, \in \mathbb{R}_+$, $\forall a \in C$, such that $\sum_{a \in C} \lambda_a > 0$ and w^* solves

$$\begin{aligned}
\max \sum_{a \in C} \lambda_a u_a(x_a) \text{ s.t. } \sum_{a \in C} x_a &\leq \sum_{a \in C} \omega_a + \sum_{a \in F} y_a \\
y_a &\in Y_b,\ b = s+1, ..., m \\
x_a &\in \mathbb{R}_+^n
\end{aligned}$$

One can show that this implies that $\forall a \in C$ there exists $\lambda_a, \in \mathbb{R}_+$, and $\varphi_a \in \mathbb{R}^n_+$, and $\widehat{p} \in \mathbb{R}^n_+$ which solve

$$\begin{aligned} \lambda_a u^*_{aj} - \widehat{p}_j + \varphi_{aj} &= 0, \ \left(\varphi_{aj} = 0 \text{ if } x^*_{aj} > 0\right) \\ \widehat{p} &\in \Gamma_b, \forall b \in F \\ \widehat{p} &\neq 0 \end{aligned}$$

where

$$\Gamma_b := \{p \in \mathbb{R}^n : py \geq pY_b\}$$

This is a well-known result in welfare analysis (see Proposition 4.3.1 of Mas-Collel (1985, p.129) for a proof).

APPENDIX

B-2

This appendix provides the proofs of Theorem 1 and 2. The notation and definitions in the previous appendix are kept. The parts, which are virtually identical to Silvestre (1985), are designated with ‡.

Lemma 10. *Let w^* be an efficient and quasi interior allocation. Then $\forall a \in C$, $\lambda_a \in \mathbb{R}_{++}$ (i.e. each consumer has a strictly positive welfare weight).*

Proof. Suppose there exists $a \in C$ such that $\lambda_a = 0$. However, condition (ii) of quasi interiority implies that there is a feasible allocation w' such that $u_b(x'_b) > u_b(x^*_b)$ for all $b \neq a$. However, this feasible allocation gives a higher value for the linear welfare program since $\sum_{a \in C} \lambda_a > 0$. This contradicts Pareto-efficiency of w^*. □

Lemma 11. *Let w^* be an Pareto-efficient allocation. Then $\widehat{p}y^*_a \geq \widehat{p}y_a$ for all $y_a \in Y_a$ and for all $a \in F$.*

Proof. Immediately follows from the fact $\widehat{p} \in \Gamma_a := \{p \in \mathbb{R}^n : py \geq pY_a\}$, $\forall a \in F$ (see Appendix 1). □

Remark 6. *These results show that consumers satisfy first order optimality conditions for individual utility maximization and firms maximize profits at the allocation w^* given prices $\widehat{p}$. Thus, in order to conclude that w^* is a Walrasian equilibrium, it is sufficient to demonstrate $\widehat{p}(x^*_a - \omega_a) \leq \sum_{b \in F} \theta^a_b \widehat{p} y^*_b$, $\forall a \in C$. Now we turn to this subject.*

Now define

$$z_a := \begin{cases} x^*_a - \omega_a \text{ if } a \in C \\ -y^*_a \text{ if } a \in F \end{cases}$$

Lemma 12. *Let w^* be efficient, quasi interior and voluntary.*
(a) If $z_{aj} > 0 > z_{ah}$ for some $a \in C \cup F$, then $\widehat{p}_j \geq \widehat{p}_h \frac{p_j}{p_h}$
(b) $\widehat{p} z_a \geq 0$, for all $a \in C$.
(c) If $y^* \neq 0$ then $y^*_h > 0$ and $\widehat{p} y^* \leq \frac{\widehat{p}_h}{p_h} p y^*$ for some commodit h.

Proof. (18.a) Let $a \in C$. When $z_{aj} > 0$ it is easy to see $x^*_{aj} > 0$. If $z_{ah} < 0$ then $x^*_{ah} > 0$ follows due to condition (i) of quasi interiority. Consequently, if $z_{aj} > 0 > z_{ah}$ then $\varphi_{aj} = \varphi_{ah} = 0$. Thus $u^*_{ah} = 0$ implies $\widehat{p}_h = 0$. But the claim is trivially true if $\widehat{p}_h = 0$. If $u^*_{ah} > 0$ then $\varphi_{aj} = \varphi_{ah} = 0$ yields

$$\frac{u^*_{aj}}{u^*_{ah}} = \frac{\widehat{p}_j}{\widehat{p}_h}.$$

On the other hand

$$u^*_{aj} - \alpha_a p_j - \overline{\gamma}_{aj} + \underline{\gamma}_{aj} + \mu_{aj} = 0$$

for all $j \in N$ implies

$$u^*_{aj} - \alpha_a p_j - \overline{\gamma}_{aj} = 0 \text{ if } z_{aj} > 0$$

and

$$u^*_{ah} - \alpha_a p_h + \underline{\gamma}_{ah} = 0 \text{ if } z_{ah} < 0$$

Since $u^*_{ah} > 0$ it must be the case that $\alpha_a > 0$. Thus

$$\frac{p_j}{p_h} = \frac{u^*_{aj} - \overline{\gamma}_{aj}}{u^*_{ah} + \underline{\gamma}_{ah}} \leq \frac{u^*_{aj}}{u^*_{ah}} = \frac{\widehat{p}_j}{\widehat{p}_h}$$

Let $a \in F$. Then $\widehat{p}_j > -\psi'_a \widehat{p}_h$ implies there is $y_a \in Y_a$ such that $\widehat{p} y_a > \widehat{p} y^*_a$ since $y^*_{aj} < 0 < y^*_{ah}$ by hypothesis. But this contradicts Pareto efficiency of w^*_a. In a similar fashion $\widehat{p}_j < -\psi'_a \widehat{p}_h$ cannot hold. Thus $\widehat{p}_j = -\psi'_a \widehat{p}_h$, $\forall a \in F$. On the other hand, $p_j > -\psi'_a p_h$ implies there exists $y_a \in Y$ such that $p y_a > p y^*_a$ and

$$\min\{0, y^*_{at}\} \leq y_{at} \leq \max\{0, y^*_{at}\}, \; t = 1, ..., n$$

we conclude $p_j \leq -\psi'_a p_h$. Since $\widehat{p}_j = -\psi'_a \widehat{p}_h$ the desired result $\widehat{p}_j \geq \widehat{p}_h \frac{p_j}{p_h}$ follows.

(18.b) Due to condition (i) of quasi interiority $\varphi_{aj} > 0$ implies $\left(\lambda_a u^*_{aj} + \varphi_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) = 0$, which gives

$$\widehat{p} z_a = \sum\nolimits_{j \in N} \left(\lambda_a u^*_{aj} + \varphi_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) = \sum\nolimits_{j \in N} \lambda_a u^*_{aj}\left(x^*_{aj} - \omega_{aj}\right)$$

However,

$$\begin{aligned}
\sum\nolimits_{j \in N} u^*_{aj}\left(x^*_{aj} - \omega_{aj}\right) &= \sum\nolimits_{j \in N} \left(\alpha_a p_j + \overline{\gamma}_{aj} - \underline{\gamma}_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) \\
&= \sum\nolimits_{j \in N} \alpha_a p_j \left(x^*_{aj} - \omega_{aj}\right) + \sum\nolimits_{j \in N} \left(\overline{\gamma}_{aj} - \underline{\gamma}_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) \\
&= \sum\nolimits_{j \in N} \alpha_a p_j \left(\sum_{b \in F} \theta^a_b p y^*_b\right) + \sum\nolimits_{j \in N} \left(\overline{\gamma}_{aj} - \underline{\gamma}_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) \\
&\geq \sum\nolimits_{j \in N} \left(\overline{\gamma}_{aj} - \underline{\gamma}_{aj}\right)\left(x^*_{aj} - \omega_{aj}\right) \geq 0
\end{aligned}$$

since $\overline{\gamma}_{aj} > 0$ implies $x^*_{aj} - \omega_a \geq 0$, and likewise, $\underline{\gamma}_{aj} > 0$ implies $x^*_{aj} - \omega_a \leq 0$. Thus $\widehat{p} z_a \geq 0$.

(18.c) Note that $y^*_j > 0$ for some j (by Lemma 16 and $0 \in Y$). Select a commodity h satisfying: $y^*_h > 0$ and $\widehat{p}_j \leq \widehat{p}_h \frac{p_j}{p_h}$ for all j such that $y^*_j > 0$. If $y^*_j > 0$ then $(\widehat{p}_j - \widehat{p}_h) \frac{p_j}{p_h} y^*_j \leq 0$. By 18.a, if $y^*_j < 0$ then $\widehat{p}_j \geq \widehat{p}_h \frac{p_j}{p_h}$, i.e., $(\widehat{p}_j - \widehat{p}_h) \frac{p_j}{p_h} y^*_j \leq 0$. Hence, $\widehat{p} y^* \leq \frac{\widehat{p}_h}{p_h} p y^*$. □

Proof of Theorem 1‡. Consider $(\widehat{p}, \mathcal{E})$ and note that

$$\begin{aligned} \sum_{a \in C} \left(\widehat{p} z_a - \sum_{b \in F} \theta^a_b \widehat{p} y^*_b \right) &= \widehat{p} \left(\sum_{a \in C} \left(z_a - \sum_{b \in F} \theta^a_b y^*_b \right) \right) \\ &= \widehat{p} \left(\sum_{a \in C} z_a - \sum_{b \in F} y^*_b \right) = 0 \end{aligned}$$

since w^* is feasible. Besides, profit maximization of y^*_b, $\forall b \in F$, for $\widehat{p}$ is ensured by Lemma 16. Since profits are zero, i.e. $p^* y^*_a = 0$, by assumption, $\widehat{p} y^*_a \leq 0$, $\forall a \in F$ due to 18.c. This implies, using Lemma 18.b, that $\widehat{p} z_a - \sum_{b \in F} \theta^a_b \widehat{p} y^*_b \geq 0$, $\forall a \in C$. Consequently, $\widehat{p} z_a - \sum_{b \in F} \theta^a_b \widehat{p} y^*_b = 0$, $\forall a \in C$. Therefore, $\forall a \in C$, x^*_a satisfies the budget constraint for $\widehat{p}$. Hence consumers optimize subject to budget constraints at w^* given prices $\widehat{p}$. This shows that w^* is a Walrasian equilibrium for $(\widehat{p}, \mathcal{E})$. □

Lemma 13. *Let $z_{aj} > 0$ (resp. $z_{aj} < 0$) for some $a \in C$. If w^* is efficient and strongly voluntary and weakly interior then $\widehat{p}_j \geq \widehat{p}_1 \frac{p_j}{p_1}$ (resp. $\widehat{p}_j \leq \widehat{p}_1 \frac{p_j}{p_1}$)*

Proof. Since w^* is strongly voluntary $\overline{\gamma}_{a1} = \underline{\gamma}_{a1} = 0$, $\forall a \in C$. Furthermore, condition (ii) of weak interiority says that $\forall (a, j) \in C \times N$, $x^*_{aj} - \omega_{aj} \neq 0$ implies

$$x^*_{b1} \left(x^*_{bj} - \omega_{bj} \right) \left(x^*_{aj} - \omega_{aj} \right) > 0$$

for some $b \in C$. Hence $z_{aj} > 0$ for some $a \in C$ implies there is $b \in C$ such that the following hold:

$$\begin{aligned} x^*_{bj} - \omega_{bj} &> 0 \\ x^*_{b1} &> 0. \end{aligned}$$

These two inequalities ensure $\underline{\gamma}_{bj} = \mu_{bj} = 0$. Furthermore, $x^*_{b1} > 0$ implies $\mu_{b1} = \varphi_{b1} = 0$. Therefore,

$$\begin{aligned} \widehat{p}_j - \widehat{p}_1 \frac{p_j}{p_1} &= \widehat{p}_j - \widehat{p}_1 \frac{u^*_{bj} - \overline{\gamma}_{aj}}{u^*_{b1} + \mu_{a1}} \\ &= \widehat{p}_j - \widehat{p}_1 \frac{u^*_{bj} - \overline{\gamma}_{aj}}{u^*_{b1}} \\ &\geq \widehat{p}_j - \widehat{p}_1 \frac{u^*_{bj}}{u^*_{b1}} = 0 \end{aligned}$$

where the last equality follows from $\widehat{p}_j = \lambda_b u^*_{bj}$ and $\widehat{p}_1 = \lambda_b u^*_{b1}$.

Now assume $z_{aj} < 0$, which ensures there is $b \in C$ such that the following hold:

$$\begin{aligned} x^*_{bj} - \omega_{bj} &< 0 \\ x^*_{b1} &> 0. \end{aligned}$$

These two inequalities ensure $\overline{\gamma}_{bj} = \mu_{bj} = 0$. Again, $x^*_{b1} > 0$ implies $\mu_{b1} = \varphi_{b1} = 0$. Thus,

$$\widehat{p}_j - \widehat{p}_1 \frac{p_j}{p_1} = \widehat{p}_j - \widehat{p}_1 \frac{u^*_{bj} + \underline{\gamma}_{bj}}{u^*_{b1}} \leq \widehat{p}_j - \widehat{p}_1 \frac{u^*_{bj}}{u^*_{b1}} = 0$$

due to $\widehat{p}_j = \lambda_b u^*_{bj}$ and $\widehat{p}_1 = \lambda_b u^*_{b1}$. This completes the proof. □

Notation 14. *Write aS_jb (read "agent a sells commodity j to agent b"), if $z_{aj} < 0$ and $z_{bj} > 0$. If aS_jb and bS_kc, write aS_jbS_kc.*

Lemma 15. *Let w^* be an Pareto-efficient, strongly voluntary, and weakly inteiror. Let $\overline{a} \in F$ satisfy: $y^*_a \neq 0$ and $\sum_{b \in B} y^*_b$ whenever $\overline{a} \in B$ for any $B \subset F$.*
21.1 There exist a consumer a^1, L commodities $\left(j^1, ..., j^L\right)$, and, if $L > 1$, $L - 1$ firms $\left(a^2, ..., a^L\right)$ such that: $a^1 S_{j^1} a^2 S_{j^2} ... a^L S_L \overline{a}$.
21.2 If $y^*_{ah} > 0$ then $\widehat{p}_h \leq \widehat{p}_l$.

Proof[‡]**.** Define $F_1 := \{\overline{a}\}$, $F_2 := F_1 \cup \{b \in F : bS_j\overline{a}$ for some $j \in N\}$, $F_{K+1} := F_K \cup \{b \in F : bS_j\overline{a}$ for some $(j, a) \in N \times F_K\}$. $K = 2, 3, ...$ Since, $\forall K$, $F_K \subset F_{K+1}$ and F is finite, $F_{\overline{K}} = F_{\overline{K}+1}$ for some $\overline{K}$. Define $\overline{y} := \sum_{b \in F_{\overline{K}}} y^*_b$. By assumption, $\overline{y} \neq 0$ and hence $\overline{y}_{j^1} < 0$ for some $j^1 \in N$. Moreover, if $b \notin F_{\overline{K}}$ then $y^*_{bj^1} \leq 0$ (otherwise $b \in F_{\overline{K}+1}$), i.e., $\sum_{b \in F} y^*_{bj^1} < 0$. Therefore, $z_{a^1 j^1} < 0$ for some $a^1 \in C$, $z_{a^2 j^1} < 0$ for some $a^2 \in F_{\overline{K}}$, and the cosntruction of $F_{\overline{K}}$ guarantees Lemma 21.1. To prove Lemma 21.2, consider the sequence of commodities $\left(j^1, ..., j^L\right)$ of Lemma 21.1. By Lemma 18.a, $\widehat{p}_h \leq \widehat{p}_{j^L} \leq \widehat{p}_{j^{L-1}} \leq ... \leq \widehat{p}_{j^1}$ and, by Lemma 19, $\widehat{p}_{j^1} \leq \widehat{p}_1$. □

Proof of Theorem 12[‡]. We only need to show $\widehat{p}z_a - \sum_{b\in F}\theta_b^a\widehat{p}y_b^* \geq 0$, $\forall a \in C$. First, Lemma 19 implies that

$$\widehat{p}z_a \geq \widehat{p}_1\frac{p^*}{p_1^*}z_a.$$

Second, if $p^*y_b^* = 0$ then $\widehat{p}y_b^* \leq 0 = \widehat{p}_h p^* y_b^*$ (by Lemma 18.c). Third, if $p^*y_b^* > 0$ then by Lemma 18.c, $\widehat{p}y_b^* \leq \widehat{p}_h p^* y_b^*$ for some h such that $y_{bh}^* > 0$, and moreover (since Lemma 21.2 holds, because $p^*\sum_{c\in B} y_c^* > 0$ and hence $p^*\sum_{c\in B} y_c^* \neq 0$ whenever $b \in B$) $\widehat{p}_h \leq \widehat{p}_1$; thus, $\widehat{p}y_b^* \leq \widehat{p}_1 p^* y_b^*$. Therefore, $\forall a \in C$, $\widehat{p}z_a - \sum_{b\in F}\theta_b^a\widehat{p}y_b^* \geq \widehat{p}_1\left(p^*z_a - \sum_{b\in F}\theta_b^a p^* y_b^*\right) = 0$, where the last equality is due to the fact that z_a satisfies the budget constraint at the price vector p^*. □